A MOCK METAPLECTIC REPRESENTATION

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Abstract. We study a unitary non irreducible representation U of a semidirect product G whose normal factor A is abelian and whose homogeneous factor H is a locally compact second countable group acting on a Riemannian manifold X. The key ingredient is a C^1 intertwining map between the actions of H on the dual group $\hat{A}$ and X. The representation U generalizes the restriction of the metaplectic representation to triangular subgroups of $Sp(d,\mathbb{R})$. For simplicity, we restrict ourselves to the case where $A=\mathbb{R}^n$ and $X=\mathbb{R}^d$. We decompose U as a direct integral and obtain necessary and sufficient conditions for its admissible vectors. Many examples are given.

Contents

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1. Introduction

Unitary representations of semidirect products have been thoroughly studied by many authors and are useful in a wide variety of applications. In particular, they play a central rôle in the harmonic analysis of the continuous wavelet transform, as discussed in [14]. From the point of view of applications, a unitary representation U of a locally compact group G (with Haar measure dg) is particularly useful if it yields a reproducing formula, that is, a weak reconstruction of the form

$$f = \int_G \langle f, U_g\eta\rangle \, U_g\eta \, dg, \tag{1}$$

valid for every f in the representation space $\mathcal{H}$, for some *admissible* vector $\eta \in \mathcal{H}$. In this case (G, U, η) is called a *reproducing* system. Alternatively, we simply say that G is a reproducing group. If U is irreducible, this is nothing else but the classical concept of square integrable representation. Typically, $\mathcal{H} = L^2(\mathbb{R}^d)$, and in this case an admissible vector η is sometimes called a *wavelet*. Apart from direct use, formula (1) is important also because it is the starting point for its discrete counterparts, an aspect that we shall not develop in the present paper. It is actually rather interesting to observe that most formulae of the above type that appear in applications, either in their continuous or discrete versions, turn out to be expressible by taking the restriction of the metaplectic representation to some triangular subgroup G of the symplectic group $Sp(d, \mathbb{R})$. This is the main theme in the papers [6], [7], and the present contribution is an outgrowth thereof.

We will be concerned with groups G that are semidirect products, where the normal factor is an abelian group A and the homogeneous factor is a locally compact second countable group H. Our main object of study is a unitary representation U of G whose construction is based on the following ingredients: a Riemannian manifold X on which H acts by C^1 diffeomorphisms and a C^1 map $\Phi : X \to \hat{A}$ (the dual group of A) that intertwines the actions of H on X and on $\hat{A}$. The representation $g \mapsto U_g$ acts on $L^2(X)$ as pointwise multiplication by the character $\Phi(x)$ if $g \in A$ and quasi regularly if $g \in H$, as clarified below in (9). For simplicity, we take $A = \mathbb{R}^n$ and $X = \mathbb{R}^d$ and we also suppose that the Jacobian of the action on $\mathbb{R}^d$ is independent of x. We call U the "mock" metaplectic representation because its definition is inspired by the case where $\mathbb{R}^n$ is a vector space of $d \times d$ symmetric matrices on which a closed subgroup H of $GL(d,\mathbb{R})$ acts by $\sigma \mapsto {}^t h^{-1}\sigma h^{-1}$. Under these circumstances, G can be identified with a triangular subgroup of $Sp(d,\mathbb{R})$ and U is the restriction to G of the metaplectic representation (see Example 1).

General admissibility criteria for type-I groups have been given in [14]. Fuhr's approach, however, assumes information that is not so easily available in many circumstances. Indeed, given the representation U on $\mathcal{H}$, his theory stems from knowledge of a direct integral decomposition $\mathcal{H} = \int_{\widehat{G}} m_\sigma \mathcal{H}_\sigma \, d\nu(\sigma)$ and a corresponding diagonalization $U = \int_{\widehat{G}} m_\sigma \sigma \, d\nu(\sigma)$. Knowing such decompositions is not a trivial task: the measure ν is known to exist, but one has to find it, together with the measurable field $\{\mathcal{H}_\sigma\}$ and the multiplicity function $\sigma \mapsto m_\sigma$. With these data at hand, Fuhr proves that if G is non-unimodular, then (1) holds true for some η if and only if ν has density with respect to $\mu_{\widehat{G}}$, the Plancherel measure of G; if G is unimodular, then one has to add the extra conditions that $m_\sigma \leq \dim \mathcal{H}_\sigma$ for ν-almost every σ and $\int_{\widehat{G}} m_\sigma \, d\nu(\sigma) < +\infty$. The explicit knowledge of $\mu_{\widehat{G}}$ is also non trivial, in general.

Without using the remarkable machinery of [14], we explicitly decompose U and thereby obtain, as a byproduct, computable admissibility criteria in terms of the intertwining map Φ.

Our finer results are Theorem 20 and Theorem 21, which deal with the cases where G is unimodular or non-unimodular, respectively. They both hold under the standard technical assumption that the H-orbits are locally closed in $\Phi(X)$ and assuming also that the H-stabilizers in $\Phi(X)$ are compact. The latter assumption may be removed and yields the weaker conclusion given in Theorem 14. Theorem 21 can actually be formulated in a very simple way by saying that U is reproducing if and only if the set of critical points of Φ has Lebesgue measure zero. This is of course very easy to check in the examples in which Φ is explicitely known.

Here is a brief ouline of the other results contained in the paper.

- Theorem 3, which establishes an important necessary condition for a reproducing formula (1) to hold true: Φ must map sets of positive measure into sets of positive measure, hence $n \leq d$. Thus we introduce an open H-invariant subset X of $\mathbb{R}^d$ with negligible Lebesgue complement whose image is denoted by $Y = \Phi(X) \subseteq \mathbb{R}^n$. The fibers $\Phi^{-1}(y)$ are Riemannian submanifolds of X and play a crucial rôle in what follows.
- Theorem 5, based on the classical coarea formula, shows how the Lebesgue measure of X disintegrates into measures ν_y concentrated on the fibers $\Phi^{-1}(y)$, whose covariance with respect to the H-action is explicitely calculated (18).
- Theorem 7, where a first reduction criterium for admissible vectors is given. One looks at the H-orbits in Y and takes their preimages under Φ in X. Upon selecting an origin y in each H-orbit in Y, one gets fibers $\Phi^{-1}(y)$ together with their H-translates in X. The theorem states that it is necessary and sufficient to test that, for almost every H-orbit in Y, the L^2-norm with respect to ν_y of any $u \in L^2(X, \nu_y)$ can be reproduced by the (weighted) H-integral of the

square modulus $|\langle u, \eta_y^h \rangle_{\nu_y}|^2$ of the components of u along the H-translates of the restriction to $\Phi^{-1}(y)$ of the admissible vector η. This is formula (21).

- Theorem 13, which exhibits a direct integral decomposition of U in terms of induced representations of isotropy subgroups of H, and is independent of any admissibility issue. This is achieved as follows.
  - First of all, we make a topological assumption, namely that the H-orbits are locally closed in Y. This is a standard assumption, without which none of the results in the current literature on these themes holds true.
  - Secondly, we derive a disintegration of the Lebesgue measure on Y *à la Mackey*, that is, $dy = \int_{\overline{Y}} \tau_{\bar{y}} \, d\lambda(\bar{y})$, where λ is a measure on the orbit space $\overline{Y} = Y/H$ and $\tau_{\bar{y}}$ is concentrated on the orbit corresponding to $\bar{y} \in \overline{Y}$. This preliminary disintegration is carried out in Theorem 8, where the covariance of $\{\tau_{\bar{y}}\}$ with respect to the H-action is also calculated (23).
  - In Proposition 10 we use the measures $\{\tau_{\bar{y}}\}$ in order to "glue" together the measures ν_y for all y in the same orbit, thereby producing new measures $\mu_{\bar{y}} = \int_Y \nu_y \, d\tau_{\bar{y}}(y)$ on X which, in turn, allow to disintegrate the Lebesgue measure on X as $dx = \int_{\overline{Y}} \mu_{\bar{y}} \, d\lambda(\bar{y})$. As before, the covariance of $\{\mu_{\bar{y}}\}$ with respect to the H-action is calculated. The reason for introducing these measures are formulae (25) and (26): the representation space of U, namely $L^2(X)$, is formally the double direct integral
  $$L^2(X) = \int_{\overline{Y}} \Big(\int_Y L^2(X, \nu_y) \, d\tau_{\bar{y}}(y) \Big) \, d\lambda(\bar{y}),$$
  where the inner integral is $L^2(X, \mu_{\bar{y}})$.
  - Next we show in Lemma 12 that $L^2(X, \mu_{\bar{y}})$ is unitarily equivalent to the representation space $\mathcal{H}_{\bar{y}}$ of the representation $W_{\bar{y}}$ which is unitarily induced to G by the quasi regular representation of the stabilizer $H_{\bar{y}}$ (naturally extended to the semidirect product $\mathbb{R}^n \rtimes H_{\bar{y}}$).

  The conclusion of Theorem 13 is that U is equivalent to $\int_{\overline{Y}} W_{\bar{y}} \, d\lambda(\bar{y})$, with an explicit intertwining isometry.

2. Notation and assumptions

In this section we fix the notation and describe the setup. We start by recalling the notions of *reproducing group* and *admissible vector*. For a thorough discussion on admissible vectors, the reader is referred to [14].

Let G be a locally compact group with (left) Haar measure dg and U be a strongly continuous unitary representation of G acting on the complex Hilbert space $\mathcal{H}$. A vector $\eta \in \mathcal{H}$ is called admissible if

$$\|f\|^2 = \int_G |\langle f, U_g \eta \rangle|^2 \, dg \qquad \text{for all } f \in \mathcal{H}.$$

If such a vector exists, we say that G is a reproducing group and that U is a *reproducing representation*. Clearly, if U is reproducing, then it is a cyclic representation, but in general it is not irreducible. When U is irreducible, the representation is reproducing if and only if it is square integrable.

2.1. **The semidirect product.** Let H be a locally compact second countable group acting on $\mathbb{R}^n$ by means of the continuous representation

$$y \mapsto h[y], \qquad h \in H. \tag{2}$$

Let G be the semidirect product $G = \mathbb{R}^n \rtimes H$ with group law

$$(a_1, h_1)(a_2, h_2) = (a_1 + h_1^\dagger[a_2], h_1 h_2) \qquad a_1, a_2 \in \mathbb{R}^n, h_1, h_2 \in H,$$

where $h^\dagger[\cdot]$ is the action given by the contragradient representation of H on $\mathbb{R}^n$ defined via the usual inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$ by

$$\langle h^\dagger[a], y \rangle = \langle a, h^{-1}[y] \rangle, \qquad a, y \in \mathbb{R}^n. \tag{3}$$

Since $h[\cdot]$ is linear, the semidirect product is well defined and G is a locally compact second countable group. Conversely, any locally compact second countable group G that is the semidirect product of a closed subgroup H and a normal subgroup V, which is a real vector space of dimension n, is of the above form.

The (left) Haar measures of G and H are written dg and dh, and, similarly, da is the Lebesgue measure on $\mathbb{R}^n$. The modular functions of G and H are denoted by Δ_G and Δ_H, respectively. The following relations are easily established

$$dg = \frac{1}{\alpha(h)} da\, dh \tag{4}$$

$$\Delta_G(a, h) = \frac{\Delta_H(h)}{\alpha(h)} \tag{5}$$

where $\alpha : H \to (0, +\infty)$ is the character of H defined by

$$\alpha(h) = |\det(a \mapsto h^\dagger[a])| = |\det(y \mapsto h^{-1}[y])|. \tag{6}$$

The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by

$$(\mathcal{F}f)(y) = \int_{\mathbb{R}^n} e^{-2\pi i \langle y, a \rangle} f(a)\, da, \qquad f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n).$$

In general, if G is any locally compact second countable group, $L^2(G)$ will denote the Hilbert space of square integrable functions with respect to left Haar measure. Finally, if X is a locally compact and second countable topological space, the Borel σ-algebra on X is denoted $\mathcal{B}(X)$ and $C_c(X)$ denotes the space of complex continuous functions on X with compact support. By *measure* we mean a σ-additive positive function μ on $\mathcal{B}(X)$ which is finite on compact sets. The hypothesis on X implies that any such measure is automatically inner and outer regular [20]. A function $f : X \to X'$ between two such spaces will be called Borel measurable if $f^{-1}(B) \in \mathcal{B}(X)$ for every $B \in \mathcal{B}(X')$ and μ-measurable if $f^{-1}(B) \in \mathcal{B}_\mu(X)$, where $\mathcal{B}_\mu(X)$ denotes the completion of $\mathcal{B}(X)$ with respect to μ.

2.2. **The mock metaplectic representation.** Suppose we are given:

(H1) a continuous action of H on $\mathbb{R}^d$ by smooth maps denoted $x \mapsto h.x$, whose Jacobian is constant; for $h \in H$ and $E \in \mathcal{B}(\mathbb{R}^d)$ we write

$$|h.E| = \beta(h)|E|; \tag{7}$$

(H2) a C^1-map $\Phi : \mathbb{R}^d \to \mathbb{R}^n$ intertwining the two actions of H, i.e.

$$\Phi(h.x) = h[\Phi(x)] \qquad x \in \mathbb{R}^d,\ h \in H. \tag{8}$$

For $g = (a, h) \in G$ we define $U_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$(U_g f)(x) = \beta(h)^{-\frac{1}{2}} e^{-2\pi i \langle \Phi(x), a\rangle} f(h^{-1}.x) \tag{9}$$

for almost every $x \in \mathbb{R}^d$.

Remarks. (a) The representation (2) of H on $\mathbb{R}^n$ plays no direct rôle in the definition of U; its purpose is to construct the semidirect product G.

(b) Occasionally, we shall write $f^h(x)$ for $f(h^{-1}.x)$.

(c) At this stage there are no limitations on the relative sizes of n and d, but we shall see later (Theorem 3) that in the situations that are of interest to us $n \leq d$.

The next Proposition records that (9) is a good definition.

Proposition 1. *The map $g \mapsto U_g$ is a strongly continuous unitary representation of G acting on $L^2(\mathbb{R}^d)$.*

Proof. Clearly, U_g is a unitary operator and U is a representation of $\mathbb{R}^n$ and H separately. In order to prove that it is a representation of G, it is enough to show that $U_h U_a U_{h^{-1}} = U_{h^\dagger[a]}$ for $a \in \mathbb{R}^n$ and $h \in H$. For $f \in L^2(\mathbb{R}^d)$, and almost every $x \in \mathbb{R}^d$

$$\begin{aligned} \left(U_h U_a U_{h^{-1}} f\right)(x) &= \beta(h)^{-\frac{1}{2}} e^{-2\pi i \langle \Phi(h^{-1}.x), a\rangle} \left(U_{h^{-1}} f\right)(h^{-1}.x) \\ &= e^{-2\pi i \langle \Phi(h^{-1}.x), a\rangle} f(x) = e^{-2\pi i \langle h^{-1}[\Phi(x)], a\rangle} f(x) \\ &= e^{-2\pi i \langle \Phi(x), h^\dagger[a]\rangle} f(x) = (U_{h^\dagger[a]} f)(x) \end{aligned}$$

To show strong continuity, it is enough to prove that $g \mapsto \langle U_g f_1, f_2\rangle$ is continuous at the identity whenever f_1, f_2 are continuous functions with compact support, and this is an easy consequence of the dominated convergence theorem. □

2.3. **Examples.** There are many interesting examples of the setup we are considering. We will focus on some situations in which most relevant features occur.

Example 1. Let H be a closed subgroup of $GL(d, \mathbb{R})$ and assume $n = d$. Since the group H acts naturally on $\mathbb{R}^d$, define

$$h.x = h[x] = h\,x \qquad x \in \mathbb{R}^d,\ h \in H.$$

Choosing $\Phi(x) = x$, the representation U is equivalent to the quasi regular representation of G via the Fourier transform. Necessary and sufficient conditions for U to be reproducing are given in [14]. It is worth observing that if G is the "$ax + b$" group, then U is

$$U_{(b,a)} f(x) = \sqrt{a} e^{-2\pi i b x} f(ax)$$

which, after conjugation with the Fourier transform, is the usual wavelet representation. It may be generalized to higher dimension (see[21]).

EXAMPLE 2. The Schrödinger representation of the Heisenberg group $\mathbb{H}^1$ may be included in this setup, by regarding $\mathbb{H}^1$ as a closed subgroup of $GL(3,\mathbb{R})$:

$$\mathbb{H}^1 = \left\{ \begin{bmatrix} 1 & q & t \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix} : q,p,t \in \mathbb{R} \right\}.$$

It is easy to see that $\mathbb{H}^1$ is the semidirect product $\mathbb{H}^1 = A \rtimes H$, where $A = \left\{ \left[\begin{smallmatrix} p \\ t \end{smallmatrix}\right] : p,t \in \mathbb{R} \right\}$ and $H = \left\{ \left[\begin{smallmatrix} 1 & 0 \\ q & 1 \end{smallmatrix}\right] : q \in \mathbb{R} \right\}$. The group H acts on $\mathbb{R}$ via translations: $q.x = x + q$ and has the natural representation on $\mathbb{R}^2$:

$$q \mapsto {}^t\begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -q \\ 0 & 1 \end{bmatrix}.$$

The smooth map $\Phi : \mathbb{R} \to \mathbb{R}^2$ defined by $\Phi(x) = \left[\begin{smallmatrix} -x \\ 1 \end{smallmatrix}\right]$ satisfies the intertwining property (8). The mock metaplectic representation takes the form

$$U_{(q,p,t)}\, f(x) = e^{-2\pi i \langle \Phi(x), \left[\begin{smallmatrix} p \\ t \end{smallmatrix}\right] \rangle} f(q^{-1}.\, x) = e^{-2\pi i (t-px)} f(x-q)$$

and it thus coincides with the Schrödinger representation, which is irreducible but notoriously not square integrable (i.e. not reproducing). Observe that $2 = n > d = 1$.

EXAMPLE 3. Let $G = \Sigma \rtimes H \subset Sp(d,\mathbb{R})$ be a triangular subgroup of the form

$$G = \left\{ \begin{bmatrix} h & 0 \\ \sigma h & {}^t h^{-1} \end{bmatrix} : h \in H,\ \sigma \in \Sigma \right\}, \tag{10}$$

where H is a closed subgroup of $GL(d,\mathbb{R})$ and Σ is an n-dimensional subspace of $\mathrm{Sym}(d,\mathbb{R})$, the space of symmetric $d \times d$ matrices.

Inner conjugation within G yields the H-action on Σ

$$h^\dagger[\sigma] := {}^t h^{-1} \sigma h^{-1} \qquad \sigma \in \Sigma, h \in H, \tag{11}$$

under which Σ must be invariant. As the notation suggests, (11) can be seen as a contragredient action. Indeed, endowing $\mathrm{Sym}(d,\mathbb{R})$ with the natural inner product $\langle \sigma_1, \sigma_2 \rangle = \mathrm{tr}(\sigma_1 \sigma_2)$, and hence Σ with its restriction denoted $\langle \cdot, \cdot \rangle_\Sigma$, and denoting by $\sigma \mapsto h[\sigma]$ the representation whose contragredient version is (11), then for $\sigma, \tau \in \Sigma$ we have

$$\langle \tau, h[\sigma] \rangle_\Sigma = \langle {}^t h \tau h, \sigma \rangle_\Sigma = \mathrm{tr}(\tau h \sigma {}^t h) = \langle \tau, P_\Sigma(h \sigma {}^t h) \rangle_\Sigma,$$

where P_Σ is the orthogonal projection from $\mathrm{Sym}(d,\mathbb{R})$ onto Σ. Thus

$$h[\sigma] = P_\Sigma(h \sigma\, {}^t h) \qquad \sigma \in \Sigma, h \in H, \tag{12}$$

and if ${}^t H = H$ there is no need of the projection.

The group H acts naturally on $\mathbb{R}^d$, that is, $h.x = hx$. Given $x \in \mathbb{R}^d$, let $\Phi(x) \in \Sigma$ be defined by

$$\mathrm{tr}(\Phi(x)\sigma) = -\frac{1}{2} \langle \sigma x, x \rangle \qquad x \in \mathbb{R}^d.$$

Identifying $\mathbb{R}^n \simeq \widehat{\Sigma} \simeq \Sigma$, we can interpret $\Phi(x)$ either as the linear functional on Σ whose action on σ is $-\frac{1}{2}\langle \sigma x, x\rangle$ or as the symmetric matrix associated to it via the usual inner product on symmetric matrices. Condition (8) is satisfied, since

$$\operatorname{tr}(\Phi(h.x)\sigma) = -\frac{1}{2}\langle {}^t h\sigma h x, x\rangle = \operatorname{tr}(\Phi(x)^t h\sigma h) = \operatorname{tr}(h\Phi(x)^t h\sigma) = \operatorname{tr}(h[\Phi(x)]\sigma).$$

The representation (9) is

$$U_{(\sigma,h)}f(x) = |\det h|^{-1/2}\, e^{\pi i\langle \sigma x, x\rangle} f(h^{-1}x). \tag{13}$$

and hence it coincides with the restriction of the metaplectic representation to the group G. Various properties of U are analyzed in [6, 7].

An important explicit example in this class is connected to the theory of shearlets [16]. Here the group G parametrizes the (two-dimensional) phase-space operations of translation, isotropic dilation and shear and is thus sometimes denoted $TDS(2)$. Precisely, $G = \mathbb{R}^2 \rtimes (\mathbb{R}_+ \times \mathbb{R})$ in the following way. The abelian normal subgroup $\Sigma \simeq \mathbb{R}^2$ consists of the 2×2 symmetric matrices $\left[\begin{smallmatrix} 0 & a_1 \\ a_1 & a_2\end{smallmatrix}\right]$. The homogeneous group $H \simeq \mathbb{R}_+ \times \mathbb{R}$ contains all the 2×2 matrices of the form $\delta^{-1/2}S_{\ell/2}$, where $\delta > 0$ and S_ℓ is the *shearing* matrix

$$S_\ell = \begin{bmatrix} 1 & \ell \\ 0 & 1\end{bmatrix},$$

so that for any $h = (\delta, \ell)$ the linear action on the abelian normal factor $\mathbb{R}^2$ is

$$h[\cdot] = \delta^{-1}\begin{bmatrix} 1 & \ell \\ 0 & 1\end{bmatrix},$$

and the group law of G is

$$(a, \delta, \ell)(a', \delta', \ell') = (a + \left[\begin{smallmatrix} \delta & 0 \\ -\delta\ell & \delta\end{smallmatrix}\right] a', \delta\delta', \ell + \ell').$$

It is easy to see that $\Phi(x_1, x_2) = -(x_1x_2, x_2^2/2)$. The mock metaplectic representation U restricted to Σ is equivalent to translations and restricted to δ it amounts to dilations, as shown in [6], where necessary and sufficient conditions for admissible vectors are given. Observe that $d = n$.

Example 4. This is a case where $n < d$. Let $H = \mathbb{R}_+ \times \mathbb{T}$. Here $\mathbb{T}$ is the one-dimensional torus, parametrized by $\theta \in [0, 2\pi)$, with Haar measure $d\theta/2\pi$, and $\mathbb{R}_+$ is the multiplicative group with Haar measure $t^{-1}dt$ where dt is the restriction to $\mathbb{R}_+$ of the Lebesgue measure on the real line. Hence H has Haar measure $dt\,d\theta/2\pi t$ and modular function $\Delta_H(h) = 1$. The representation of H on $\mathbb{R}$ is

$$h[y] = t^2 y \qquad y \in \mathbb{R},$$

where $h = (t, \theta)$. Hence in particular $\alpha(h) = t^{-2}$. The group law in $G = \mathbb{R} \rtimes H$ is

$$(a_1, t_1, \theta_1)(a_2, t_2, \theta_2) = (a_1 + t_1^{-2}a_2, t_1t_2, \theta_1 + \theta_2).$$

The resulting Haar measure is $tdt\,d\theta/2\pi$ and the modular function $\Delta_G(a, t, \theta) = t^2$. The action of $h = (t, \theta) \in H$ on $\mathbb{R}^2$ is given by

$$h.(x_1, x_2) = t(\cos\theta\, x_1 - \sin\theta\, x_2, \sin\theta\, x_1 + \cos\theta\, x_2) \qquad (x_1, x_2) \in \mathbb{R}^2$$

so that $\beta(h) = t^2$. Finally, $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is given by $\Phi(x_1, x_2) = x_1^2 + x_2^2$. The mock metaplectic representation $U$of G on $L^2(\mathbb{R}^2)$ is

$$U_{(a,t,\theta)} f(x_1, x_2) = t^{-1} e^{-2\pi i (x_1^2 + x_2^2) a} f\left(t^{-1}(\cos\theta\, x_1 + \sin\theta\, x_2), t^{-1}(-\sin\theta\, x_1 + \cos\theta\, x_2)\right).$$

EXAMPLE 5. Let $H = \mathbb{R}^* \times \mathbb{R}$ where $\mathbb{R}^*$ is the (non-connected) multiplicative group of non-zero real numbers and $\mathbb{R}$ is the additive group with Haar measuress $|t|^{-1}dt$ and db respectively. The Haar measure of H is $|t|^{-1}dtdb$ and $\Delta_H = 1$. An element $h = (t, b) \in H$ acts on $\mathbb{R}$ and $\mathbb{R}^2$ by means of

$$h[y] = ty \qquad y \in \mathbb{R}$$

$$h.(x_1, x_2) = (x_1 + b, tx_2) \qquad (x_1, x_2) \in \mathbb{R}^2$$

so that $\alpha(h) = |t|^{-1}$ and $\beta(h) = |t|$. Finally $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is defined by $\Phi(x_1, x_2) = x_2$, which clearly satisfies (8).

3. MAIN RESULTS

3.1. **Dimensional constraints.** Our first result, Theorem 3, states that if G is reproducing, then $n \leq d$. The interpretation of this statement in the case of wavelets is that the dimension of the space of translations cannot exceed that of the "ground" space. In order to prove the theorem we need a technical lemma, in the proof of which we use a standard result in harmonic analysis on locally compact abelian groups (see Theorem (31.33) in [18]). This is the fact that if a bounded measure ν on the locally compact abelian group $\mathcal{G}$ has Fourier transform that coincides almost everywhere (on the character group $\widehat{\mathcal{G}}$) with the Fourier transform of an $L^p(\mathcal{G})$-function F, with $1 \leq p \leq 2$, then $F \in L^1(\mathcal{G})$, ν is absolutely continuous with respect to Haar measure and its Radon-Nikodym derivative is F. We apply this to a bounded measure on $\mathbb{R}^n$.

LEMMA 2. *For any $f, \eta \in L^2(\mathbb{R}^d)$ the following facts are equivalent:*

(i) $\int_G |\langle f, U_g \eta\rangle|^2\, dg < +\infty$*;*
(ii) *for almost every $h \in H$ the bounded measure on $\mathbb{R}^n$*

$$\Omega_h(E) = \int_{\Phi^{-1}(E)} f(x)\overline{\eta(h^{-1}.x)}\, dx, \qquad E \in \mathcal{B}(\mathbb{R}^n), \tag{14}$$

has a density $\omega_h \in L^2(\mathbb{R}^n)$.

Under the above circumstances

$$\int_G |\langle f, U_g \eta\rangle|^2\, dg = \int_H \left(\int_{\mathbb{R}^n} |\omega_h(y)|^2\, dy\right) \frac{dh}{\alpha(h)\beta(h)}. \tag{15}$$

Proof. Observe that Ω_h is the image measure, induced by Φ, of the bounded measure with density $f\overline{\eta^h} \in L^1(\mathbb{R}^d)$ with respect to dx (see e.g. Sec. 39 in [17]). Since Ω_h is bounded, the basic integration formula for image measures, (see Theorem C, p.161 in [17]) and (9) imply that

$$\langle f, U_{(a,h)}\eta\rangle = \beta^{-\frac{1}{2}}(h) \int_{\mathbb{R}^d} e^{2\pi i \langle \Phi(x), a\rangle} f(x)\overline{\eta^h}(x)\, dx = \beta^{-\frac{1}{2}}(h) \int_{\mathbb{R}^n} e^{2\pi i \langle y, a\rangle}\, d\Omega_h(y).$$

Assume that $\int_G |\langle f, U_g \eta\rangle|^2\, dg < \infty$. Since $dg = \frac{da\, dh}{\alpha(h)}$, Fubini's theorem implies that, for almost every $h \in H$,

$$\int_{\mathbb{R}^n} |\langle f, U_{(a,h)}\eta\rangle|^2\, da = \beta(h)^{-1} \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{2\pi i\langle y,a\rangle}\, d\Omega_h(y)|^2\, da < +\infty.$$

This says that the Fourier transform of Ω_h is in $L^2(\mathbb{R}^n)$, and the aforementioned Theorem (31.33) in [18] ensures that the latter condition is equivalent to saying that Ω_h has an $L^2(\mathbb{R}^n)$-density ω_h with respect to dy. Furthermore, by Plancherel

$$\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{2\pi i\langle y,a\rangle}\, d\Omega_h(y)|^2\, da = \int_{\mathbb{R}^n} |\omega_h(y)|^2\, dy.$$

Applying again Fubini's theorem, (15) is proved. Therefore (i) implies (ii). The converse statement is shown by applying the same argument backwards. □

We are now in a position to state our first result.

Theorem 3. *If U is a reproducing representation, then the image under Φ of any Borel subset of $\mathbb{R}^d$ with positive measure has positive measure. In particular,*

(i) *$n \le d$;*
(ii) *the set of critical points*[1] *of Φ is an H-invariant subset of $\mathbb{R}^d$ of measure zero.*

Proof. By contradiction, suppose that there exists a Borel subset A of $\mathbb{R}^d$ with positive measure such that $\Phi(A)$ is negligible. Since $|A|_d > 0$ and the Lebesgue measure is regular, there exists a compact subset $K \subset A$ with $|K|_d > 0$. Clearly, $\Phi(K)$ is also compact, but $|\Phi(K)|_n = 0$. Take an admissible vector η for U. The reproducing formula for $f = \chi_K$ and (15) imply that

$$0 < |K|_d = \int_H \left(\int_{\mathbb{R}^n} |\omega_h(y)|^2\, dy\right) \frac{dh}{\alpha(h)\beta(h)},$$

so that, on a subset of H of positive Haar measure we have $\omega_h \neq 0$. Take then $h \in H$ such that $\Omega_h = \omega_h dy \neq 0$. Now, if E is a Borel subset of $\mathbb{R}^n$, the definition of Ω_h gives

$$\Omega_h(E) = \Omega_h(E \cap \Phi(K)) = \int_{E\cap\Phi(K)} \omega_h(y)dy = 0$$

because $|\Phi(K)|_n = 0$. Hence $\Omega_h = 0$, a contradiction.

To show (i), assume that $n > d$ and apply the above result to $A = \mathbb{R}^d$. Since Φ is of class C^1 we have $|\Phi(A)|_n = 0$, so that U cannot be reproducing.

To show (ii), denote by $\mathcal{C}$ the set of critical points of Φ. Sard's theorem implies that $\Phi(\mathcal{C})$ has measure zero. But then, by (i), also $\mathcal{C}$ has measure zero. Finally, H-invariance of $\mathcal{C}$ will follow from

$$\Phi_{*h.x}(h_*.v) = h[\Phi_{*x}v], \qquad x, v \in \mathbb{R}^d, \tag{16}$$

where h_* denotes the differential of the action $x \mapsto h.x$ and is therefore linear. Indeed, (16), together with the linearity of $u \mapsto h[u]$, shows that $v \in \ker\Phi_{*x}$ if and only $h_*.v \in \ker\Phi_{*h_*.x}$, so that $\dim\ker\Phi_{*x} = \dim\ker\Phi_{*h.x}$. Since $x \in \mathcal{C}$ if and only if $\dim\ker\Phi_{*x} > d - n$, the claim follows. To prove (16), fix $x \in \mathbb{R}^d$, a tangent vector

[1]A point $x \in \mathbb{R}^d$ is critical for $\Phi : \mathbb{R}^d \to \mathbb{R}^n$ if the rank of the differential map Φ_{*x} is less than n.

$v \in T_x(\mathbb{R}^d) \simeq \mathbb{R}^d$ and a smooth curve $v(t)$ passing through x at time zero with tangent vector v. Evidently, $h.v(t)$ is smooth and has tangent $h_*.v$ at time zero. By (8) and again by the linearity of $u \mapsto h[u]$

$$\Phi_{*h.x}(h_*.v) = \frac{d}{dt}\Phi(h.v(t))\Big|_{t=0} = \frac{d}{dt}h[\Phi(v(t))]\Big|_{t=0} = h[\frac{d}{dt}\Phi(v(t))\Big|_{t=0}] = h[\Phi_{*x}v],$$

as desired. □

3.2. **Measures concentrated on the preimages under Φ.** Given any $x \in \mathbb{R}^d$, let $J(\Phi)(x) = \sqrt{\det(\Phi_{*x} \cdot {}^t\Phi_{*x})}$ be the Jacobian of Φ at x and denote by $\mathcal{R}$ the set of regular points of Φ, namely

$$\mathcal{R} = \left\{x \in \mathbb{R}^d : J(\Phi)(x) > 0\right\}.$$

LEMMA 4. *The set $\mathcal{R}$ satisfies the following properties:*

(i) *it is open;*
(ii) *it is H-invariant and has H-invariant image;*
(iii) *the restriction of Φ to it is an open mapping;*
(iv) *for every y in its image, the fiber $\Phi^{-1}(y)$ is a Riemannian submanifold of $\mathbb{R}^d$.*

Proof. (i) Since Φ has continuous derivatives, $\mathcal{R}$ is an open set. (ii) The H-invariance follows from the fact that $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{C}$ and (ii) of Theorem 3. The H-invariance of the image follows from (8). Finally, (iii) and (iv) are standard consequences of the fact that, by definition of $J(\Phi)$, the differential Φ_{*x} is surjective whenever $x \in \mathcal{R}$. □

ASSUMPTION 1. Motivated by Theorem 3, in the following we assume that $\mathcal{C}$ (the complement of $\mathcal{R}$) has Lebesgue measure zero. In particular, we assume that $n \leq d$. Furthermore, we fix an open H-invariant subset X of $\mathcal{R}$ whose complement also has measure zero and we denote by Y its image under Φ, namely $Y = \Phi(X)$. Clearly, X satisfies the properties (i)–(iv) described in Lemma 4 and has full measure.

The next results are based on several kinds of disintegration formulae and their covariance properties with respect to the H-action. In Section 5.1 we review the general theory of disintegration of measures and introduce the pertinent notation. As for the induced H-action on measures, and the resulting covariance properties, we recall that, if ν is a measure on X and $h \in H$, ν^h is the measure given by $\nu^h(E) = \nu(h.E)$ whenever $E \in \mathcal{B}(X)$. Equivalently,

$$\int_X f(x)\, d\nu^h(x) = \int_X f(h^{-1}.x)\, d\nu(x) \tag{17}$$

for every $f \in C_c(X)$. The first disintegration we discuss arises from the Coarea Formula for submersions.

THEOREM 5. *There exists a unique family $\{\nu_y\}$ of measures on X, labeled by the points of Y, with the following properties:*

(i) *ν_y is concentrated on $\Phi^{-1}(y)$ for all $y \in Y$;*
(ii) *$dx = \int_Y \nu_y dy$;*

(iii) *for any* $\varphi \in C_c(X)$ *the map* $y \mapsto \int_X \varphi(x)\, d\nu_y(x) \in \mathbb{C}$ *is continuous.*

Furthermore,

$$\nu^h_{h[y]} = \alpha(h)\beta(h)\, \nu_y \tag{18}$$

for all $h \in H$ *and all* $y \in Y$*.*

Proof. The proof is based on the classical Coarea Formula. In Section 5.3 we give a short proof adapted to the situation at hand and we introduce the notation used also in this proof. The reader is thus referred to Theorem 27 below.

Define ν_y by (58). Property (i) is then obvious and (ii) is the content of Theorem 27.

To prove (iii), fix $\varphi \in C_c(X)$ and $y_0 \in Y$. If $y_0 \notin \Phi(\operatorname{supp}\varphi)$, there is an open neighborhood V of y_0 such that $V \cap \Phi(\operatorname{supp}\varphi) = \emptyset$. Thus $\int_X \varphi(x)\, d\nu_y(x) = 0$ for all $y \in V$. If $y_0 \in \Phi(\operatorname{supp}\varphi)$, taking a finite covering if necessary, we can always assume that there exists a diffeomorphism $\Psi : U \times V \mapsto W$ such that (60) holds, where U is an open subset of $\mathbb{R}^{d-n}$, V is an open neighborhood of y_0 and W is an open subset of X containing $\operatorname{supp}\varphi$. The definition of ν_y gives

$$\int_X \varphi(x)\, d\nu_y(x) = \int_U \varphi(\Psi(z,y))(J\Psi)(z,y)\, dz$$

and the map $y \mapsto \int_U \varphi(\Psi(z,y))(J\Psi)(z,y)\, dz$ is continuous on V by the dominated convergence theorem.

In order to show (18), fix $h \in H$. Given $\varphi \in C_c(X)$ and $\xi \in C_c(Y)$, apply both sides of the equality in (ii) to the function $x \mapsto \xi(\Phi(x))\varphi(x)$

$$\begin{aligned}
\int_X \xi(\Phi(x))\varphi(x)\, dx &= \int_Y \xi(y) \left(\int_X \varphi(x)\, d\nu_y(x) \right) dy \\
(x \mapsto h.x) \quad &= \int_Y \xi(y) \left(\int_X \varphi(h.x)\, d\nu^h_y(x) \right) dy \\
(y \mapsto h[y]) \quad &= |\det h[\cdot]| \int_Y \xi(h[y]) \left(\int_X \varphi(h.x)\, d\nu^h_{h[y]}(x) \right) dy.
\end{aligned}$$

Since $|\det h[\cdot]| = \alpha(h)^{-1}$, we get

$$\int_X \xi(\Phi(x))\varphi(x)\, dx = \alpha(h)^{-1} \int_Y \xi(h[y]) \left(\int_X \varphi(h.x)\, d\nu^h_{h[y]}(x) \right) dy.$$

Applying first the change of variable $x \mapsto h.x$ and then again (ii), we also have

$$\begin{aligned}
\int_X \xi(\Phi(x))\varphi(x)\, dx &= \beta(h) \int_X \xi(\Phi(h.x))\varphi(h.x)\, dx \\
&= \beta(h) \int_Y \xi(h[y]) \left(\int_X \varphi(h.x)\, d\nu_y(x) \right) dy.
\end{aligned}$$

Hence, since these equalities hold for every $\xi \in C_c(Y)$ and $\varphi \in C_c(X)$, it follows that

$$\int_X \varphi(h.x)\, d\nu^h_{h[y]}(x) = \alpha(h)\beta(h) \int_X \varphi(h.x)\, d\nu_y(x), \qquad \text{a.e. } y \in Y.$$

Replacing φ with φ^h yields

$$\int_X \varphi(x)\, d\nu^h_{h[y]}(x) = \alpha(h)\beta(h) \int_X \varphi(x)\, d\nu_y(x) \qquad y \notin Y_0,$$

where $Y_0 \subset Y$ is a set of Lebesgue measure zero, possibly depending on φ. We claim that the above equality actually holds true for all $y \in Y$, independently of φ. Indeed, take a sequence $(y_n)_{n\in\mathbb{N}}$ in $Y \setminus Y_0$ converging to y. By (iii)

$$\begin{aligned}\int_X \varphi(x)\, d\nu^h_{h[y]}(x) &= \lim_{n\to\infty} \int_X \varphi(x)\, d\nu^h_{h[y_n]}(x)\\ &= \lim_{n\to\infty} \alpha(h)\beta(h) \int_X \varphi(x)\, d\nu_{y_n}(x)\\ &= \alpha(h)\beta(h) \int_X \varphi(x)\, d\nu_y(x).\end{aligned}$$

Since ν_y is a measure, (18) follows. □

In view of the previous result, we may apply the theory developed in Section 5.2. In particular we obtain (56) in the case in which ω and ρ are the Lebesgue measures:

$$L^2(X) = \int_Y L^2(X, \nu_y) dy, \qquad f = \int_Y f_y dy. \tag{19}$$

Here the equalities must be interpreted in $M(X)$ and the second integral is a scalar integral relative to the duality of $M(X)$ and $C_c(X)$. For a discussion of the details see the Appendix, where it is also explained that in particular

$$\|f\|^2 = \int_Y \|f_y\|^2_{\nu_y}\, dy. \tag{20}$$

One of the reasons for introducing the measures $\{\nu_y\}$ is because, via the coarea formula, they provide a very useful description of the density ω_h discussed in Lemma 2.

COROLLARY 6. *Given $f, \eta \in L^2(X)$, the function $y \mapsto \langle f_y, \eta^h_y\rangle_{\nu_y}$ coincides almost everywhere with the density ω_h of the measure Ω_h defined by (14).*

Proof. Item (iii) of Theorem 25, together with Theorem 5, applied to $f\bar{\eta} \in L^1(X)$ and any $\xi \in C_c(Y)$ gives

$$\int_X \xi\left(\Phi(x)\right) f(x)\bar{\eta}(h^{-1}.x)\, dx = \int_Y \xi(y) \int_X f(x)\bar{\eta}(h^{-1}.x)\, d\nu_y(x)\, dy.$$

The left hand side is nothing else but the integral $\int_Y \xi(y)\, d\Omega_h(y)$ because Ω_h is the image measure, induced by Φ, of $f\overline{\eta^h}\, dx$. The corollary follows. □

3.3. Reduction to fibers.

Much of our analysis stems from decomposing the representation space $L^2(\mathbb{R}^d)$ in terms of the measures $\{\nu_y\}$, and from a rather detailed understanding of the H-action on Y. We thus introduce the usual notation for group actions: if $y \in Y$, then H_y is the stabilizer of y, $H[y] = \{h[y] : h \in H\}$ is the corresponding orbit and Y/H the orbit space. At this stage we therefore need a hypothesis ensuring that the Y/H is not a pathological measurable space. It is worth mentioning that this hypothesis is satisfied in all the significant examples that we are aware of. Below we further comment on this.

ASSUMPTION 2. We assume that for every $y \in Y$ the H-orbit $H[y]$ is locally closed in Y, i.e., that it is open in its closure or, equivalently, that $H[y]$ is the intersection of an open and a closed set.

The above assumption is not enough to ensure that the orbit space Y/H is a Hausdorff space, hence locally compact, with respect to the quotient topology. However, it is possible to bypass this topological obstruction by choosing a different parametrization of the H-orbits of Y. Indeed, a result of Effros (Theorem 2.9 in [11]) shows that Assumption 2 is equivalent to the fact that the orbit space Y/H is a standard Borel space. Hence there is a locally compact second countable space Z and a Borel measurable (hence Lebesgue measurable) map $\pi : Y \to Z$ such that $\pi(y) = \pi(y')$ if and only if y and y' belong to the same orbit. In the following, we fix the space Z, whose points will label the orbits of Y, and we choose on Z a pseudo-image measure[2] λ of the Lebesgue measure under the map π. We note that λ is concentrated on $\pi(Y)$ and a subset E is λ-negligible if and only if $\pi^{-1}(E)$ is a negligible subset of Y, which is equivalent to the fact that $(\pi \circ \Phi)^{-1}(E)$ is negligible subset of $\mathbb{R}^d$.

Theorem 7. *Under Assumption 1, the following conditions are equivalent:*

(i) *the vector $\eta \in L^2(\mathbb{R}^d)$ is admissible for U;*
(ii) *for λ-almost every $\bar{y} \in Z$, there exists a point $y \in \pi^{-1}(\bar{y})$ such that*

$$\|u\|^2_{\nu_y} = \int_H |\langle u, \eta_y^h \rangle_{\nu_y}|^2 \frac{dh}{\alpha(h)\beta(h)}, \qquad u \in L^2(X, \nu_y). \tag{21}$$

If (21) *holds true for y, then it holds true for every point in $H[y]$.*

Proof. On the one hand, U is reproducing if and only if there exists a square integrable η such that for every $\varphi \in C_c(X)$

$$\int_G |\langle \varphi, U_g \eta \rangle|^2 \, dg = \int_X |\varphi(x)|^2 \, dx = \int_Y \int_X |\varphi(x)|^2 \, d\nu_y(x) \, dy,$$

the latter being a consequence of the coarea formula (20). On the other hand, by Lemma 2, U is reproducing if and only if the measure Ω_h defined by (14) has an L^2-density ω_h for almost every $h \in H$ and formula (15) holds true, and Corollary 6 tells us that ω_h can be expressed in terms of the measures $\{\nu_y\}$. Therefore

$$\begin{aligned}
\int_Y \int_X |\varphi(x)|^2 \, d\nu_y(x) \, dy &= \int_G |\langle \varphi, U_g \eta \rangle|^2 \, dg \\
&= \int_H \left(\int_Y |\omega_h(y)|^2 \, dy \right) \frac{dh}{\alpha(h)\beta(h)} \\
&= \int_H \int_Y |\langle \varphi, \eta_y^h \rangle_{\nu_y}|^2 \, dy \, \frac{dh}{\alpha(h)\beta(h)} \\
&= \int_Y \int_H |\langle \varphi, \eta_y^h \rangle_{\nu_y}|^2 \, \frac{dh}{\alpha(h)\beta(h)} \, dy,
\end{aligned}$$

where in the last line we have applied Fubini's theorem. Reasononing as in the proof of Corollary 6, the equality of the first and last terms of the above string is equivalent to saying that (21) holds for almost every $y \in Y$. Next, we show that if (21) holds

[2] It is a measure on Z whose sets of measure zero are exactly the sets whose preimage with respect to π have measure zero in Y. It always exists since Y is σ-compact: it is enough to take first a finite measure on Y equivalent to the Lebesgue measure (just choose a positive L^1 density), and then to consider the image measure on Z induced by π (see e.g. Chap. VI, Sect. 3.2 in [3]).

for a given $y \in Y$, then it holds for every point in its orbit $H[y]$. Take any $h \in H$. Using (17) and (18), and assuming (21) for y, we obtain

$$\begin{aligned}
\int_X |\varphi(x)|^2 \, d\nu_{h[y]}(x) &= \int_X |\varphi(h.x)|^2 \, d\nu^h_{h[y]}(x) \\
&= \int_X |\varphi(h.x)|^2 \, \alpha(h)\beta(h) d\nu_y(x) \\
&= \alpha(h)\beta(h) \int_H \left| \int_X \varphi(h.x)\bar{\eta}(k^{-1}.x) \, d\nu_y(x) \right|^2 \frac{dk}{\alpha(k)\beta(k)} \\
(h.x = z) \quad &= \alpha(h)\beta(h) \int_H \left| \int_X \varphi(z)\bar{\eta}((hk)^{-1}.z) \, d\nu_y^{h^{-1}}(z) \right|^2 \frac{dk}{\alpha(k)\beta(k)} \\
(hk = \ell) \quad &= \alpha^2(h)\beta^2(h) \int_H \left| \int_X \varphi(z)\bar{\eta}(\ell^{-1}.z) \, d\nu_y^{h^{-1}}(z) \right|^2 \frac{d\ell}{\alpha(\ell)\beta(\ell)} \\
&= \int_H \left| \int_X \varphi(z)\bar{\eta}(\ell^{-1}.z)\alpha(h)\beta(h) \, d\nu^{h^{-1}}_{h^{-1}(h[y])}(z) \right|^2 \frac{d\ell}{\alpha(\ell)\beta(\ell)} \\
&= \int_H \left| \int_X \varphi(z)\bar{\eta}(\ell^{-1}.z) \, d\nu_{h[y]}(z) \right|^2 \frac{d\ell}{\alpha(\ell)\beta(\ell)} \\
&= \int_H |\langle \varphi, \eta^\ell \rangle_{\nu_{h[y]}}|^2 \, \frac{d\ell}{\alpha(\ell)\beta(\ell)},
\end{aligned}$$

that is (21) for $h[y]$, as desired. Hence (21) holds on a union of orbits that fills out a subset Z of Y whose complement Z' has measure zero. Now, Z' is also H-invariant and it projects onto $\pi(Z')$, whose measure is zero because such is the measure of $\pi^{-1}(\pi(Z')) = Z'$. In principle, Z could depend on φ. We show next that this is not the case.

As explained in the Appendix (see Footnote 4), there exists a sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(X)$ with the following property: given an arbitrary $\varphi \in C_c(X)$, there exists a compact set $K \subset X$ and a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ such that

$$\text{(22)} \qquad \operatorname{supp}(\varphi) \subset K, \qquad \operatorname{supp}(\varphi_{n_k}) \subset K, \qquad \lim_{k\to\infty} \sup_{x\in X} |\varphi_{n_k}(x) - \varphi(x)| = 0.$$

Let Z_n denote the (H-invariant and conull) subset of Y on which (21) holds for φ_n. By dominated convergence and (22), we find that (21) holds for any $\varphi \in C_c(X)$ and any y in the intersection $Z_\infty := \cap_n Z_n$, which is obviously H-invariant and whose complement $Y \setminus Z_\infty = \cup_n (X \setminus Z_n)$ has measure zero. □

Note. Since π induces a Borel isomorphism between the orbit space Y/H and $\Phi(Y)$, in the above statement and in the theorems of the following section, it would be possible to avoid the space Z by considering on Y/H a σ-finite measure defined on the quotient σ-algebra, which, by Assumption 2 (Theorem 2.9 in [11]) coincides with the Borel σ-algebra induced by the quotient topology. However, this measure could fail to be finite on compact subsets.

3.4. **Disintegration formulae.** Our next result, Theorem 14, is based on some classical formulae that allow both a geometric interpretation of the integral (21) and a computational reduction that in the known examples is indeed significant. This is inspired by the irreducible case, where it is known that U is reproducing (i.e. square integrable) if and only if the H-orbit, unique by irreducibility, has full measure and the inducing representation of the stabilizer H_y is square integrable as well [1].

We allude to formulae that express an integral over Y as a double integral, first along the single H-orbits and then with respect to the measure λ on the space Z. Although these kinds of formulae can be traced back to Bourbaki [4] and Mackey [22], perhaps one of the most famous occurences of such a disintegration procedure appears in the celebrated paper of Kleppner and Lipsman [19]; for a recent review see [15]. Much in the same spirit, we shall also need to decompose integrals over H by integrating along a closed subgroup H_0 first, and then over the homogeneous space H/H_0, which we identify with a suitable orbit of Y. The topological hypothesis A2 is needed in order that these decomposition formulae can be safely applied.

Recall that in the beginning of Section 3.3 we fixed a space Z labelling the orbits of Y and a measure λ on Z whose null sets are in one-to-one correspondence with the H-invariant null sets of Y.

THEOREM 8. *There exists a λ-full subset $\overline{Y}$ of Z and a family $\{\tau_{\bar{y}}\}$ of measures on Y, labeled by the points of $\overline{Y}$, with the following properties:*

(i) *$\tau_{\bar{y}}$ is concentrated on $\pi^{-1}(\bar{y})$ for all $\bar{y} \in \overline{Y}$;*
(ii) *$dy = \int_{\overline{Y}} \tau_{\bar{y}}\, d\lambda(\bar{y})$.*

Furthermore, for all $h \in H$ and all $\bar{y} \in \overline{Y}$

$$\tau_{\bar{y}}^h = \alpha(h)^{-1}\tau_{\bar{y}}. \tag{23}$$

The family $\{\tau_{\bar{y}}\}$ is unique in the sense that if $\{\tau'_{\bar{y}}\}$ is another family satisfying (i) *and* (ii)*, then $\tau'_{\bar{y}} = \tau_{\bar{y}}$ for almost every $\bar{y} \in \overline{Y}$.*

Proof. The content of the theorem can be found in many different papers, see Lemmas 11.1 and 11.5 of [22] and Theorem 2.1 of [19], in slightly different contexts. The cited results are both based on Bourbaki's treatment of disintegration of measures. Here we simply adapt this theory to our setting.
Theorem 2 Ch.VI § 3.3 of [3] yields a family $\{\tau_z\}$ of measures on X labeled by the points $z \in Z$, and unique in the sense of the statement, such that

- $\tau_z \neq 0$ if and only if $z \in \pi(Y)$
- τ_z is concentrated on $\pi^{-1}(z)$
- $dy = \int_Z \tau_z dz$.

We now show that for almost every $z \in \pi(Y)$ the measure τ_z is relatively invariant with respect to the action of H (see also Lemmas 11.3 and 11.5 of [22]). Fix $h \in H$. By definition $(dy)^h = \alpha(h^{-1})dy$. For all $\varphi \in C_c(Y)$,

$$\int_Y \varphi(y)\alpha(h^{-1})dy = \int_Z \left(\int_Y \varphi(y)\alpha(h^{-1})d\tau_z(y)\right) dz.$$

The left hand side is also equal to

$$\int_Y \varphi(h^{-1}[y])\,dy = \int_Z \Big(\int_Y \varphi(h^{-1}[y]) d\tau_z(y)\Big)\,dz = \int_Z \Big(\int_Y \varphi(y) d\tau_z^h(y)\Big)\,dz,$$

so that $\int_Z \alpha(h^{-1})\tau_z dz = \int_Z \tau_z^h dz$, and both τ_z^h and $\alpha(h^{-1})\tau_z$ are concentrated on $\pi^{-1}(z)$. Hence, by iv) of Theorem 25, $\tau_z^h = \alpha(h^{-1})\tau_z$ for almost every $z \in Z$. The set of z such that the above equality holds is actually independent of h. Indeed, since H is second countable, there is a dense countable subset $\mathcal{D} \subset H$ and, by the above equality, a negligible set N such that $\tau_z^h = \alpha(h^{-1})\tau_z$ for all $h \in \mathcal{D}$, whenever $z \notin N$. Take $h \notin \mathcal{D}$. By density there is a sequence (h_n) in $\mathcal{D}$ such that h_n converges to h. For every $z \notin N$ and every $\varphi \in C_c(X)$

$$\begin{aligned}\int_Y \varphi(y)\, d\tau_z^h(y) &= \lim_n \int_Y \varphi(y) d\tau_z^{h_n}(y)\\ &= \lim_n \int_Y \varphi(y)\alpha(h_n^{-1})(x) d\tau_z(y) = \int_Y \varphi(y)\alpha(h^{-1}) d\tau_z(y),\end{aligned}$$

so that $\tau_z^h = \alpha(h^{-1})\tau_z$. Since $\tau_z \neq 0$ if $z \in \pi(Y)$, the measure τ_z is relatively invariant. □

Assumption 2 is needed in order to apply the cited theorem from [3]. The same theorem actually holds under the (weaker) conditions that are described in the lemma below. Their equivalence does not seem to be a known fact. In [15], Theorem 12, it is shown that (i) is a necessary condition for the disintegration formula (ii) of Theorem 8 to hold true. In the statement below $\hat{\pi}$ denotes the canonical projection from Y onto Y/H.

LEMMA 9. *The following two conditions are equivalent:*

- *(i) there exists an increasing sequence of compact subset $\{K_n\}$ of Y such that the complement of $\cup K_n$ is Lebesgue negligible and $\hat{\pi}(K_n)$ is an Hausdorff space, endowed with the relative topology*[3]*;*
- *(ii) there exists an H-invariant null set $N \subset Y$ such that $(Y \setminus N)/H$ is a standard Borel space.*

Proof. First we show that (i) implies (ii). Denote by R the equivalence relation induced by the action of H on Y, that is, $y \sim_R y'$ if and only if $\hat{\pi}(y) = \hat{\pi}(y')$. Taking into account that X is σ-compact and footnote 3, (i) states that R is a Lebesgue measurable equivalence relation according to the definition in Ch. VI § 3.4 of [3]. Hence Proposition 2 Ch. VI § 3.4 of [3] implies there exists a locally compact second countable space Z and a Lebesgue measurable map $p : Y \to Z$ such that $p(y) = p(y')$ if and only if $\hat{\pi}(y) = \hat{\pi}(y')$. Egoroff theorem implies there exists an increasing sequence of compact subsets, denoted again by $\{K_n\}$ such that the complement of $\cup K_n$ is Lebesgue negligible and the restriction of p to K_n is continuous. Without loss of generality we can suppose that $\overline{p(Y)} \neq Z$ and we fix $z_0 \notin \overline{p(Y)}$. For any n define $p_n : Y \to Z$ as $p_n(y) = p(y')$ if $y = h[y']$ for some $h \in H$ and $y' \in K_n$, and

[3] Due to Prop. 3 Ch.1 § 5.3 of [2], $\hat{\pi}(K_n)$ is a Hausdorff space if and only if the quotient space K_n/R_n is Hausdorff with respect to the quotient topology where R_n is the equivalence relation induced by H on K_n.

$p_n(y) = z_0$, otherwise (observe that if $y = h'[y'']$ for some $h' \in H$ and $y'' \in K_n$, then $\hat{\pi}(y') = \hat{\pi}(y'')$ so that $p(y') = p(p''))$. We claim that p_n is Borel measurable. First we show that for any closed subset $C \subset Z$ with $z_0 \notin C$, $p_n^{-1}(C)$ is a Borel set. By construction $p_n^{-1}(C) = H[p_{K_n}^{-1}(C)]$ where $p_{K_n}^{-1}(C)$ is a closed subset of K_n since p_{K_n} is continuous, hence it is a compact subset of K_n since K_n is compact and, a fortiori, a compact subset of Y. Since H is σ-compact and the action of H on Y is continuous, $H[p_{K_n}^{-1}(C)]$ is a countable union of compact subsets, hence it is Borel set. As a special case, $H[K_n]$ is also a Borel set as well as its complement $p_n^{-1}(z_0)$. It follows that $p_n^{-1}(C)$ is a Borel set for any closed subset $C \subset Z$, so that p_n is Borel measurable. The complement N of $\cup_n H[K_n]$ is an H-invariant Borel subset of Y with zero Lebesgue measure. By construction, for any $n < m$ clearly $p_m(y) = p_n(y)$ for all $y \in K_n \subset K_m$, so that $\lim_n p_n(y) = p(y)$ exists for all $y \notin N$ and the restriction of p to the H-invariant Borel set $Y \setminus N$ is Borel measurable. Hence there is an injective Borel map $j : (Y \setminus N)/H \to Z$, that is, $(Y \setminus N)/H$ is a standard Borel subset.

To show the converse result, let N as in (ii). Since $(Y \setminus N)/H$ is a standard Borel space, there exists a Borel injective map $i : (Y \setminus N)/H \to \mathbb{R}$. Furthermore, fix any section $s : N/H \to N$ and define $p : Y \to Y \times \mathbb{R}$

$$p(y) = \begin{cases} (y_0, i(\hat{\pi}(y))) & y \notin N \\ (s(\hat{\pi}(y)), 0) & y \in N \end{cases},$$

where $y_0 \in Y \setminus N$. Clearly the map p is Lebesgue measurable and $p(y') = p(y)$ if and only if $\hat{\pi}(y) = \hat{\pi}(y')$. Egoroff theorem implies there exists an increasing sequence of compact subsets $\{K_n\}$ such that the complement of $\cup K_n$ is Lebesgue negligible and the restriction of p to K_n is continuous. A standard result of topology ensures that $\hat{\pi}(K_n)$ is homeomorphic to $p(K_n)$ which is a compact subset of an Hausdorff space, so it is Hausdorff too. □

3.5. **An integral decomposition of U.** In this section Assumptions 1 and 2 are taken for granted. The main result of this section is that Theorems 5 and 8, which hold both true, provide an integral decomposition of the mock metaplectic representation in terms of induced representations of isotropy subgroups of H. This result, which is of some independent interest, is at the root of Theorem 14, which characterizes the admissible vectors for U.

PROPOSITION 10. *Possibly redefining $\overline{Y}$ by subtracting a negligible set if necessary, for every $\bar{y} \in \overline{Y}$ the family of measures $\{\nu_y\}$ is scalarly integrable with respect to $\tau_{\bar{y}}$, the measure on X*

$$\mu_{\bar{y}} = \int_Y \nu_y \, d\tau_{\bar{y}}(y)$$

is concentrated on the H-invariant closed subset $\Phi^{-1}(\pi^{-1}(\bar{y}))$ and for all $h \in H$

$$\mu_{\bar{y}}^h = \beta(h)\mu_{\bar{y}}.$$

Furthermore, the family of measures $\{\mu_{\bar{y}}\}$ is scalarly integrable with respect to λ and

$$dx = \int_{\overline{Y}} \mu_{\bar{y}} \, d\lambda(\bar{y}).$$

Proof. The coarea formula, that is (ii) of Theorem 5, says that if f is an integrable function with respect to the Lebesgue measure, then also $y \mapsto \langle \nu_y, f\rangle = \int_X f(x) d\nu_y(x)$, is Lebesgue integrable, and

$$\int_Y \langle \nu_y, f\rangle \, dy = \int_X f(x)\, dx.$$

Theorem 8 allows us to apply (iii) of Theorem 25 to $\langle \nu_y, f\rangle$ and we thus have a negligible set $N \subset \overline{Y}$ such that, $\langle \nu_y, f\rangle$ is integrable with respect to $\tau_{\bar{y}}$ for all $\bar{y} \notin N$, the map $\bar{y} \mapsto \int_Y \langle \nu_y, f\rangle \, d\tau_{\bar{y}}(y)$ is integrable with respect to λ and, by what we have just established

$$\int_{\overline{Y}} \left(\int_Y \langle \nu_y, f\rangle \, d\tau_{\bar{y}}(y) \right) d\lambda(\bar{y}) = \int_Y \langle \nu_y, f\rangle \, dy = \int_X f(x)\, dx. \tag{24}$$

Fix an increasing sequence (U_k) of relatively compact subsets covering X with $\overline{U_k} \subset U_{k+1}$. For each k, the above argument applied to the characteristic function of U_k implies the existence of a negligible set $N_k \subset \overline{Y}$ such that, for all $\bar{y} \notin N_k$, the map $y \mapsto \nu_y(U_k)$ is integrable with respect to $\tau_{\bar{y}}$. Fix a countable subset $\mathcal{S}$ of $C_c(X)$ such that, for any $\varphi \in C_c(X)$, there is a sequence (φ_i) in $\mathcal{S}$ converging to φ uniformly and $|\varphi_i| \leq |\varphi|$ for all i. For each i, the above result applied to $\varphi \in \mathcal{S}$ implies the existence of a negligible set $N_\varphi \subset \overline{Y}$ such that, for all $\bar{y} \notin N_\varphi$, the map $y \mapsto \int_X \varphi, d\nu_y(x)$ is integrable with respect to $\tau_{\bar{y}}$.
Put $N = (\cup_{k\in\mathbb{N}} N_k) \cup (\cup_{\varphi\in\mathcal{S}} N_\varphi) \subset \overline{Y}$, a λ-negligible set. We now claim that the family ν_y is scalarly integrable with respect to $\tau_{\bar{y}}$ for all $\bar{y} \notin N$. Indeed, given $\varphi \in C_c(X)$, there is a sequence (φ_i) in $\mathcal{S}$ converging to φ uniformly and $|\varphi_i| \leq |\varphi|$ for all i, and an index k such that $\operatorname{supp}\varphi \subset U_k$. The dominated convergence theorem implies that the sequence $\langle \nu_y, \varphi_i\rangle$ converges pointwise to $\langle \nu_y, \varphi\rangle$. Furthermore,

$$|\langle \nu_y, \varphi_i\rangle| \leq \int_X |\varphi(x)| \, d\nu_y \leq \|\varphi\|_\infty \nu_y(U_k) \qquad \text{for all } y \in Y.$$

By construction, if $\bar{y} \notin N$, $y \mapsto \nu_y(U_k)$ and all the function $y \mapsto \langle \nu_y, \varphi_i\rangle$ are integrable with respect to τ_y, so that also $y \mapsto \langle \nu_y, \varphi\rangle$ is $\tau_{\bar{y}}$-integrable by dominated convergence. The claim is proved and, by definition of $\mu_{\bar{y}}$,

$$\int_X \varphi(x)\, d\mu_{\bar{y}}(x) = \int_Y \langle \nu_y, \varphi\rangle(y)\, d\tau_{\bar{y}}(y).$$

Equality (24) applied to φ implies that the function $\bar{y} \mapsto \int_Y \langle \nu_y, \varphi\rangle \, d\tau_{\bar{y}}(y)$, which is almost everywhere defined, is λ-integrable, so that the family $\{\mu_{\bar{y}}\}$ is scalarly-integrable, and $dx = \int_{\overline{Y}} \mu_{\bar{y}} \, d\lambda(\bar{y})$, that is

$$\int_X \varphi(x)\, dx = \int_{\overline{Y}} \left(\int_X \varphi(x)\, d\mu_{\bar{y}} \right) d\lambda(\bar{y}).$$

By the intertwining property of Φ, the closed set $\Phi^{-1}(\pi^{-1}(\bar{y}))$ is H-invariant. Put now $A_k = U_k \setminus \Phi^{-1}(\pi^{-1}(\bar{y}))$. If $\bar{y} \notin N$, then since ν_y is concentrated on $\Phi^{-1}(y)$, and $\Phi^{-1}(y) \cap A_k = \emptyset$, and since $\tau_{\bar{y}}$ is concentrated on $\pi^{-1}(\bar{y})$, for $\tau_{\bar{y}}$-almost every $y \in Y$

$$\mu_{\bar{y}}(A_k) = \int_Y \nu_y(A_k)\, d\tau_{\bar{y}}(y) = 0.$$

Hence, given $h\in H$ and $\varphi\in C_c(X)$

$$\begin{aligned}\int_X \varphi(h^{-1}.x)d\mu_{\bar y} &= \int_Y\Big(\int_X \varphi(h^{-1}.x)\,d\nu_y(x)\Big)\,d\tau_{\bar y}(y)\\ &=\alpha(h)\beta(h)\int_Y\Big(\int_X\varphi(x)\,d\nu_{h^{-1}[y]}(x)\Big)\,d\tau_{\bar y}(y)\\ &=\beta(h)\int_Y\Big(\int_X\varphi(x)\,d\nu_y(x)\Big)\,d\tau_{\bar y}(y)=\beta(h)\int_X\varphi(x)d\mu_{\bar y}\end{aligned}$$

where the second line is due to the change of variables $x\mapsto h.x$ and (18), and the third line is due to the change of variables $y\mapsto h.y$ and (23). ∎

From now on we regard the measure λ as restricted to the set $\overline{Y}$ defined in the above proposition. Furthermore, Theorem 26, or equation (56), yields the following identifications as Hilbert spaces

$$L^2(X)=\int_{\overline{Y}} L^2(X,\mu_{\bar y})\,d\lambda(\bar y) \qquad f=\int_{\overline{Y}} f_{\bar y}\,d\lambda(\bar y) \tag{25}$$

$$L^2(X,\mu_{\bar y})=\int_Y L^2(X,\nu_y)\,d\tau_{\bar y}(y) \qquad f_{\bar y}=\int_Y f_{\bar y,y}\,d\tau_{\bar y}(y), \tag{26}$$

where $f\in L^2(X)$, $f_{\bar y}\in L^2(X,\mu_{\bar y})$ for all $\bar y\in\overline{Y}$ and, fixed $\bar y$, $f_{\bar y,y}\in L^2(X,\nu_y)$ for all $y\in Y$. The integrals of Hilbert spaces are direct integrals with respect to the measurable field associated with $C_c(X)$, and the integral of functions are scalar integrals of vector valued functions taking value in $M(X)$ by regarding $L^2(X)$, $L^2(X,\mu_{\bar y})$ and $L^2(X,\nu_y)$ as a subspace of $M(X)$ in a natural way. In particular, if $f\in C_c(X)$, $f_{\bar y}$ is the restriction of f to $\Phi^{-1}(\pi^{-1}(\bar y))$ and $f_{\bar y,y}$ is the restriction to $\Phi^{-1}(y)$. Furthermore, for any $f\in L^2(X)$

$$\|f\|^2=\int_{\overline{Y}}\int_Y \|f_{\bar y,y}\|^2_{\nu_y}\,d\tau_{\bar y}(y)\,d\lambda(\bar y). \tag{27}$$

Lemma 11. *Given $y\in Y$ and $h\in H$, the operator $T_{y,h}:L^2(X,\nu_y)\to L^2(X,\nu_{h[y]})$*

$$(T_{y,h}f)(x)=\sqrt{\alpha(h^{-1})\beta(h^{-1})}f(h^{-1}.x) \qquad \nu_{h[y]}\text{-almost every } x\in X$$

is unitary. Furthermore, for every $h,h'\in H$ and every $y\in Y$

$$T_{h[y],h'}T_{y,h}=T_{y,h'h} \tag{28}$$

$$T_{y,h}^{-1}=T_{h[y],h^{-1}}. \tag{29}$$

Proof. Given a Borel measurable function f which is square-integrable with respect to ν_y, the map $x\mapsto(T_{y,h}f)(x)$ is also Borel measurable and it is square-integrable with respect to $\nu_{h[y]}$ since

$$\alpha(h^{-1})\beta(h^{-1})\int_X|f(h^{-1}.x)|^2\,d\nu_{h[y]}(x)=\int_X|f(x)|^2\,d\nu_y(x),$$

by the change of variables $x\mapsto h.x$ and (18). The above equation implies that $T_{y,h}$ is a well-defined isometry from $L^2(X,\nu_y)$ to $L^2(X,\nu_{h[y]})$. Equality (28) is clear and, as a consequence, $T_{h[y],h^{-1}}T_{y,h}=T_{y,e}$ is the identity on $L^2(X,\nu_{h[y]})$ so that $T_{y,h}$ is surjective, thereby showing (29). ∎

For any $\bar{y} \in \overline{Y}$, we fix an origin y_0 in the orbit $\pi^{-1}(\bar{y}) = H[y_0]$ and we denote by $H_{\bar{y}}$ the stabilizer at y_0. We denote by $\mathcal{K}_{\bar{y}} = L^2(X, \nu_{y_0})$ and by $\Lambda_{\bar{y}}$ the quasi-regular representation of $H_{\bar{y}}$ acting on $\mathcal{K}_{\bar{y}}$ whose value at $s \in H_{\bar{y}}$ is $\Lambda_{\bar{y},s} = T_{y_0,s}$. As usual, we extend $\Lambda_{\bar{y}}$ to a representation of $\mathbb{R}^n \rtimes H_{\bar{y}}$ by setting $\Lambda_{\bar{y},a} = e^{-2\pi i \langle y_0, a\rangle}\,\mathrm{id}$ for all $a \in \mathbb{R}^n$. Finally, we denote by $W_{\bar{y}}$ the representation of G unitarily induced by $\Lambda_{\bar{y}}$ from $\mathbb{R}^n \rtimes H_{\bar{y}}$ to G. We realize $W_{\bar{y}}$ as a representation acting on the space $\mathcal{H}_{\bar{y}}$ of those functions $F : G \to \mathcal{K}_{\bar{y}}$ that satisfy

(K1) F is dg-measurable;
(K2) For all $g \in G$ and $(a, s) \in \mathbb{R}^n \rtimes H_{\bar{y}}$

$$F(gas) = \sqrt{\alpha(s^{-1})}\; \Lambda^{-1}_{\bar{y},as}\, F(g);$$

(K3) $\|F\|^2_{\mathcal{H}_{\bar{y}}} := \int_Y \|F(h(y))\|^2_{\mathcal{K}_{\bar{y}}}\, \alpha(h(y)) d\tau_{\bar{y}}(y) < +\infty$.

Here $h(y) \in H$ is any element in H that satisfies $h(y)[y_0] = y$. Since $\tau_{\bar{y}}$ is concentrated on $H[y_0]$ it is enough to define $h(y)$ for $y \in H[y_0]$ and, due to the covariance property in (K2), the integral does not depend on the choice of $h(y)$ in the coset $hH_{\bar{y}}$. Two functions F and F' are identified if $\|F - F'\|^2_{\mathcal{H}_{\bar{y}}} = 0$. The induced representation on $\mathcal{H}_{\bar{y}}$ is defined by the equality

$$(W_{\bar{y},g}F)(g') = F(g^{-1}g')$$

valid for dg-almost every $g' \in G$.

LEMMA 12. *Fix $\bar{y} \in \overline{Y}$. The map $S_{\bar{y}}$ whose value at $f_{\bar{y}} = \int_Y f_{\bar{y},y}\, d\tau_{\bar{y}}(y)$ is given by*

$$(S_{\bar{y}} f_{\bar{y}})(a, h) = \sqrt{\alpha(h^{-1})}\; e^{2\pi i \langle h[y_0], a\rangle}\; T^{-1}_{y_0,h}(f_{\bar{y},h[y_0]})$$

is a unitary operator from $L^2(X, \mu_{\bar{y}})$ onto $\mathcal{H}_{\bar{y}}$.

Proof. For any $(a, h) \in G$, $f_{\bar{y},h[y_0]} \in L^2(X, \nu_{h[y_0]})$. Hence $T^{-1}_{y_0,h}(f_{\bar{y},h[y_0]}) \in \mathcal{K}_{\bar{y}}$. In order to prove that $S_{\bar{y}} f_{\bar{y}}$ is dg-measurable it is enough to show that

$$h \mapsto \langle T^{-1}_{y_0,h}(f_{\bar{y},h[y_0]}), \varphi\rangle_{\mathcal{K}_{\bar{y}}} = \int_X f_{\bar{y},h[y_0]}(h^{-1}.x)\varphi(x) d\nu_{y_0}(x)$$

is dh-measurable for every $\varphi \in C_c(X)$ because $C_c(X)$ is a dense subspace of the separable Hilbert space $\mathcal{K}_{\bar{y}}$. Without loss of generality we assume that $f_{\bar{y}}$ is a positive function and, as a consequence, also $f_{\bar{y},y} \geq 0$ for all $y \in Y$. Hence $\omega_y = f_{\bar{y},y} \cdot \nu_y$ is a measure in X. By assumption, the map $y \mapsto \omega_y$ is scalarly integrable with respect to $\tau_{\bar{y}}$. Since X is second countable, Prop. 2 of Ch.5 § 3.1 in [3] and Lusin's theorem ensure that there exists an increasing sequence of compact subsets K_n such that $Y \setminus (\cup K_n)$ is $\tau_{\bar{y}}$-negligible and for all n and $\varphi \in C_c(X)$ the map $y \mapsto \int_X \varphi(x) d\omega_y(x)$ restricted to K_n is continuous. Put

$$C_n = \{h \in H \mid h[y_0] \in K_n\}.$$

Then $H \setminus (\cup C_n)$ is dh-negligible since the measure $\tau_{\bar{y}}$ is concentrated on the orbit $\pi^{-1}(\bar{y}) = H[y_0]$ and $\tau_{\bar{y}}$ is a non-zero measure which is relatively invariant with respect to the action of H. Clearly, for all n and $\varphi \in C_c(X)$ the map $h \mapsto \int_X \varphi(x) d\omega_{h[y_0]}(x)$ restricted to C_n is continuous.

Now, $h \mapsto \varphi^h$ is continuous from H into $C_c(X)$. We claim that for all n the map $h \mapsto \int_X \varphi(h^{-1}.x)d\omega_{h[y_0]}(x)$ restricted to C_n is continuous. Indeed, since H is metrizable, it is enough to consider a sequence (h_k) in C_n converging to $h \in C_n$. Setting $y_k = h_k[y_0]$, $y = h[y_0]$, $\varphi_k = \varphi^{h_k}$ and $\omega_k = \omega_{y_k}$, $\omega = \omega_y$, we get

$$\begin{aligned}|\int_X \varphi_k(x)d\omega_n(x) - \int_X \varphi^h(x)d\omega(x)| &\leq |\int_X (\varphi_k(x) - \varphi^h(x))d\omega_n(x)| \\ &\quad + |\int_X \varphi^h(x)d\omega_n(x) - \int_X \varphi^h(x)d\omega(x)| \\ &\leq \sup_{x\in K}|\varphi_k(x) - \varphi^h(x)| \sup_{n\in\mathbb{N}} \omega_n(K) \\ &\quad + |\int_X \varphi^h(x)d\omega_n(x) - \int_X \varphi^h(x)d\omega(x)|,\end{aligned}$$

where $\sup_{n\in\mathbb{N}} \omega_n(K)$ is finite by the Banach-Steinhaus theorem, and the last two summands go to zero by construction. Hence, the map $h \mapsto \int_X \varphi(h^{-1}.x)d\omega_{h[y_0]}(x)$ is the limit almost everywhere of a sequence of measurable functions, and so it is dh-measurable.

Next we prove the covariance property (K2). For $g = (a, h) = ah$ and $(b, s) = bs \in G_{\bar{y}}$,

$$\begin{aligned}(S_{\bar{y}}f_{\bar{y}})(ahbs) = (S_{\bar{y}}f_{\bar{y}})(a + h^\dagger[b], hs) &= \sqrt{\alpha(h^{-1})\alpha(s^{-1})}\; e^{2\pi i\langle hs[y_0], a+h^\dagger[b]\rangle}\; T^{-1}_{y_0,hs}(f_{\bar{y},hs[y_0]}) \\ &= \sqrt{\alpha(s^{-1})}\; e^{2\pi i\langle h[y_0], h^\dagger[b]\rangle}\; T^{-1}_{y_0,s}(S_{\bar{y}}f_{\bar{y}})(a, h) \\ &= \sqrt{\alpha(s^{-1})}\; e^{2\pi i\langle y_0, b\rangle}\; \Lambda^{-1}_{y_0,s}(S_{\bar{y}}f_{\bar{y}})(a, h)\end{aligned}$$

by definition of $h^\dagger$ and $\Lambda_{\bar{y}}$. Further,

$$\begin{aligned}\int_Y \|(S_{\bar{y}}f_{\bar{y}})(h(y))\|^2_{\mathcal{K}_{\bar{y}}}\, \alpha(h(y))d\tau_{\bar{y}}(y) &= \int_Y \|T^{-1}_{y_0,h(y)}(f_{\bar{y},h(y)[y_0]})\|^2_{\mathcal{K}_{\bar{y}}} d\tau_{\bar{y}}(y) \\ &= \int_Y \|f_{\bar{y},y}\|^2_{\nu_y} d\tau_{\bar{y}}(y) = \int_X |f(x)|^2 d\mu_{\bar{y}}(x),\end{aligned}$$

whence (K3). This also shows that $S_{\bar{y}}$ is an isometry from $L^2(X, \mu_{\bar{y}})$ into $\mathcal{H}_{\bar{y}}$.

Finally we prove that $S_{\bar{y}}$ is surjective. Given $F \in \mathcal{H}_{\bar{y}}$, for all $h \in H$ define

$$f_{\bar{y},h} = \sqrt{\alpha(h)}\; T_{y_0,h}(F(h)) \in L^2(X, \nu_{h[y_0]}).$$

Since F satisfies (K2), it follows that $f_{\bar{y},hs} = f_{\bar{y},h}$. For $\varphi \in C_c(X)$ the map

$$h \mapsto \sqrt{\alpha(h)}\langle T_{y_0,h}(F(h)), \varphi\rangle_{\nu_{h[y_0]}} = \sqrt{\alpha(h)}\langle F(h), \varphi^h\rangle_{\mathcal{K}_{\bar{y}}}$$

is dh-measurable since $h \mapsto F(h)$ is dh-measurable from H into $\mathcal{K}_{\bar{y}}$ and $h \mapsto \sqrt{\alpha(h)}\varphi^h$ is continuous from H into $\mathcal{K}_{\bar{y}}$. Therefore

$$\int_Y \|f_{\bar{y},h(y)}\|^2_{\nu_y}\, d\tau_{\bar{y}}(y) = \int_Y \|F(h(y))\|^2_{\mathcal{K}_{\bar{y}}}\, \alpha(h(y))d\tau_{\bar{y}}(y) < +\infty.$$

It follows that $f_{\bar{y}} = \int_Y f_{\bar{y},h(y)}\, d\tau_{\bar{y}}(y)$ is in $\int_Y L^2(X, \nu_y)\, d\tau_{\bar{y}}(y) = L^2(X, \mu_{\bar{y}})$ and, by construction $S_{\bar{y}}f_{\bar{y}} = F$. ∎

Recall that $L^2(X) = \int_Y L^2(X, \mu_{\bar{y}})\, d\lambda(\bar{y})$, where the direct integral is defined by the measurable structure associated with any fixed dense countable family $\{\varphi_k\}$ in $C_c(X)$.

Clearly, $\bar{y} \mapsto \{S_{\bar{y}}\varphi_k\}$ is a measurable structure for the family $\{\mathcal{H}_{\bar{y}}\}$, and we define the direct integral $\mathcal{H} = \int_{\overline{Y}} \mathcal{H}_{\bar{y}}\, d\lambda(\bar{y})$.

THEOREM 13. *The map* $S : L^2(X) \to \mathcal{H}$

$$Sf = \int_{\overline{Y}} S_{\bar{y}} f_{\bar{y}}\, d\lambda(y) \qquad f = \int_{\overline{Y}} f_{\bar{y}}\, d\lambda(y)$$

is a unitary map intertwining the mock metaplectic representation U *with the unitary representation* W *of* G *acting on* $\mathcal{H}$ *given by*

$$W = \int_{\overline{Y}} W_{\bar{y}}\, d\lambda(\bar{y}).$$

Proof. The only fact we need to prove is the intertwining property, which we can verify on the dense subset $C_c(X) \subset L^2(X)$. We observe that

$$\varphi = \int_{\overline{Y}} \varphi_{\bar{y}} d\lambda(\bar{y}) \qquad \varphi_{\bar{y}} = \int_{Y} \varphi_{\bar{y},y} d\tau_{\bar{y}}(y)$$

where $\varphi_{\bar{y}} = \varphi \in L^2(X, \mu_{\bar{y}})$ and $\varphi_{\bar{y},y} = \varphi \in L^2(X, \nu_y)$. For any $g \in G$, U_g leaves $C_c(X)$ invariant so that it is enough to prove that for every $\bar{y} \in \overline{Y}$

$$(S_{\bar{y}}(U_g\varphi))(h) = (S_{\bar{y}}\varphi)(g^{-1}h).$$

for almost every $h \in G$; but since two functions in $\mathcal{H}_{\bar{y}}$ that are equal for almost all $h \in H$, they are equal almost everywhere in G due to the covariance property (K2), the above equality needs only been proved for almost every $h \in H$. If $g = a \in \mathbb{R}^n$

$$\begin{aligned}
(S_{\bar{y}}(U_a\varphi))(h) &= \sqrt{\alpha(h^{-1})}\, T_{y_0,h}^{-1} \left(e^{-2\pi i \langle \Phi(\cdot), a\rangle}\varphi\right) \\
&= \sqrt{\alpha(h^{-1})}\, e^{-2\pi i \langle h[y_0], a\rangle} T_{y_0,h}^{-1}\varphi \\
&= (S_{\bar{y}}\varphi)(-a, h) = (S_{\bar{y}}\varphi)(a^{-1}h)
\end{aligned}$$

where in the second line, $\Phi(x) = h[y_0]$ for $\nu_{h[y_0]}$-almost all $x \in X$. If $g = k \in H$,

$$\begin{aligned}
(S_{\bar{y}}(U_k\varphi))(h) &= \sqrt{\alpha(h^{-1})}\, T_{y_0,h}^{-1} \left(\sqrt{\beta(k^{-1})}\varphi^k\right) \\
&= \sqrt{\alpha(h^{-1})}\sqrt{\alpha(k)} T_{y_0,h}^{-1} (T_{k^{-1}h[y_0],k}\varphi) \\
&= \sqrt{\alpha((k^{-1}h)^{-1})} \left(T_{k^{-1}h[y_0],k}^{-1} T_{y_0,h}\right)^{-1} \varphi \\
&= \sqrt{\alpha((k^{-1}h)^{-1})} \left(T_{h[y_0],k^{-1}} T_{y_0,h}\right)^{-1} \varphi \\
&= \sqrt{\alpha((k^{-1}h)^{-1})} \left(T_{y_0,k^{-1}h}\right)^{-1} \varphi = (S_{\bar{y}}\varphi)(k^{-1}h).
\end{aligned}$$

□

3.6. **Admissible vectors.** We are at last in a position to state our main result. We need, however, a last disintegration formula, sometimes referred to as the Mackey-Bruhat formula (see e.g. [13]), a rather straightforward consequence of the theory of quasi-invariant measures on homogeneous spaces. The easiest way of formulating it is perhaps that for any $\varphi \in C_c(H)$ the following integral formula holds

$$\int_H \varphi(h)\alpha(h^{-1})dh = \int_Y \left(\int_{H_{\bar{y}}} \varphi(h(y)s)ds \right) d\tau_{\bar{y}}(y), \tag{30}$$

where ds is a fixed Haar measure on the stabilizer $H_{\bar{y}}$ and where as before $h(y) \in H$ is any element that satisfies $h(y)[y_0] = y$ for $\tau_{\bar{y}}$-almost every $y \in Y$. We interpret (30) along the same lines of thought that we have followed for the other formulae by writing

$$\alpha^{-1} \cdot dh = \int_Y (ds)^{h(y)^{-1}} d\tau_{\bar{y}}(y) \tag{31}$$

as an equality of measures on H. This time ds is regarded as a measure on H concentrated on $H_{\bar{y}}$, so that the translated measure $(ds)^{h(y)^{-1}}$ is concentrated on $h(y)H_{\bar{y}}$. As usual, we shall extend (30) to L^1-functions by means of Theorem 25.

Theorem 13 establishes that U and W are equivalent. Therefore, we formulate our necessary and sufficient condition for the existence of admissible vectors of U for those of W. Thus, any admissible vector $F \in \mathcal{H}$ for W is to be thought of as the image under $S : L^2(X) \to \mathcal{H}$ of an analyzing wavelet η.

Theorem 14. *Under Assumptions 1 and 2, $F = \int F_{\bar{y}}\, d\lambda(\bar{y})$ is an admissible vector for W if and only if for almost every $\bar{y} \in \overline{Y}$ and for every $u \in \mathcal{K}_{\bar{y}}$*

$$\|u\|^2_{\mathcal{K}_{\bar{y}}} = \int_Y \left(\int_{H_{\bar{y}}} |\langle u, \Lambda_{\bar{y},s} \left(F_{\bar{y}} \Delta_G^{-1/2} \right) (h(y)) \rangle_{\mathcal{K}_{\bar{y}}}|^2 \, ds \right) \alpha(h(y))\, d\tau_{\bar{y}}(y). \tag{32}$$

Proof. By the definition of T given in Lemma 11, the equality

$$T_{y_0,h^{-1}}(\eta^h)_{\bar{y},y_0}(x) = \sqrt{\alpha(h)\beta(h)}(\eta^h)_{\bar{y},h^{-1}[y_0]}(h.x) = \sqrt{\alpha(h)\beta(h)}\eta_{\bar{y},h^{-1}[y_0]}(x)$$

holds for any $\eta \in L^2(X)$ and hence

$$(\eta^h)_{\bar{y},y_0} = \sqrt{\alpha(h)\beta(h)}\, (T_{y_0,h^{-1}})^{-1} \eta_{\bar{y},h^{-1}[y_0]}.$$

Suppose now that η is an admissible vector for U or, equivalently, that $F = S\eta$ is such for W. By Theorem 7, what we have just established and the definition of S given in

Lemma 12, we obtain

$$
\begin{aligned}
\|u\|^2_{\mathcal{K}_{\bar y}} &= \int_H |\langle u, (\eta^h)_{\bar y, y_0}\rangle|^2 \frac{dh}{\alpha(h)\beta(h)} \\
&= \int_H |\langle u, \sqrt{\alpha(h)\beta(h)}\,(T_{y_0,h^{-1}})^{-1}\,\eta_{\bar y, h^{-1}[y_0]}\rangle|^2 \frac{dh}{\alpha(h)\beta(h)} \\
&= \int_H |\langle u, S_{\bar y}\eta_{\bar y}(h^{-1})\rangle|^2 \frac{dh}{\alpha(h)} \\
&= \int_H |\langle u, F_{\bar y}(h^{-1})\rangle|^2 \frac{dh}{\alpha(h)} \\
(h \mapsto h^{-1}) \quad &= \int_H |\langle u, F_{\bar y}(h)\rangle|^2 \Delta_H(h^{-1})\alpha(h)\, dh.
\end{aligned} \tag{33}
$$

Now, by Theorem 2 (and the comments below) in Ch. VII § 3.5 of [4], for all $s \in H_{\bar y}$ the modular functions of H and $H_{\bar y}$ are related by the formula

$$
\alpha^{-1}(s) = \frac{\Delta_{H_{\bar y}}(s)}{\Delta_H(s)}. \tag{34}
$$

Hence, applying (30), the covariance property (K2), (34) and (5) we obtain

$$
\begin{aligned}
\|u\|^2_{\mathcal{K}_{\bar y}} &= \int_Y \left(\int_{H_{\bar y}} |\langle u, F(h(y)s)\rangle|^2 \frac{\alpha^2(h(y)s)}{\Delta_H(h(y)s))}\, ds \right) d\tau_{\bar y}(y) \\
&= \int_Y \left(\int_{H_{\bar y}} |\langle u, \sqrt{\alpha(s^{-1})}\Lambda_{\bar y, s^{-1}} F(h(y))\rangle|^2 \frac{\alpha^2(h(y)s)}{\Delta_H(h(y)s))}\, ds \right) d\tau_{\bar y}(y) \\
&= \int_Y \left(\int_{H_{\bar y}} |\langle u, \Lambda_{\bar y, s^{-1}} F(h(y))\rangle|^2 \frac{\alpha^2(h(y))}{\Delta_H(h(y))} \Delta_{H_{\bar y}}(s^{-1})\, ds \right) d\tau_{\bar y}(y) \\
(s \mapsto s^{-1}) \quad &= \int_Y \left(\int_{H_{\bar y}} |\langle u, \Lambda_{\bar y, s} F(h(y))\rangle|^2 \frac{1}{\Delta_G(h(y))}\, ds \right) \alpha(h(y)) d\tau_{\bar y}(y),
\end{aligned}
$$

which is (32). Conversely, if (32) holds for some $F \in \mathcal{H}$, then reading the above strings of equalities backwards yields the first line in (33). Therefore, by Theorem 7, η is admissible for U, hence F is such for W. ∎

COROLLARY 15. *Assume that U is a reproducing representation and suppose that y is a point in an orbit $\bar y$ for which* (32) *holds true.*

(i) If $\Phi^{-1}(y)$ is a finite set, then the stabilizer H_y is compact;
(ii) If G is unimodular and the stabilizer H_y is compact, then $\Phi^{-1}(y)$ is a finite set, hence $n = d$.

Proof. Clearly, it is enough to prove (i) and (ii) for the origin y_0. Take a (countable) Hilbert basis $\{u_i\}$ of $\mathcal{K}_{\bar{y}}$. Apply (32) to each element of the basis and sum

$$
\begin{aligned}
\dim \mathcal{K}_{\bar{y}} &= \int_Y \Big(\int_{H_{\bar{y}}} \sum_i |\langle u_i, \Lambda_{\bar{y},s} \left(F_{\bar{y}} \Delta_G^{-1/2} \right) (h(y)) \rangle_{\mathcal{K}_{\bar{y}}}|^2 \, ds \Big) \alpha(h(y)) \, d\tau_{\bar{y}}(y) \\
&= \int_Y \Big(\int_{H_{\bar{y}}} \| \Lambda_{\bar{y},s} \left(F_{\bar{y}} \Delta_G^{-1/2} \right) (h(y)) \|^2_{\mathcal{K}_{\bar{y}}} \, ds \Big) \alpha(h(y)) \, d\tau_{\bar{y}}(y) \\
&= \int_{H_{\bar{y}}} ds \cdot \int_Y \| F_{\bar{y}} \Delta_G^{-1/2} (h(y)) \|^2_{\mathcal{K}_{\bar{y}}} \alpha(h(y)) \, d\tau_{\bar{y}}(y).
\end{aligned}
\tag{35}
$$

Now, if $\Phi^{-1}(y)$ is a finite set, then the left hand side is finite and strictly positive, hence so is the right hand side, so $H_{\bar{y}}$ has finite volume. This proves (i). If $\Delta_G = 1$ and $H_{\bar{y}}$ has finite volume, then the right hand side is finite and strictly positive by (K3). Hence $\Phi^{-1}(y)$ is a finite set and since it is a regular submanifold of dimension $d - n$, necessarily $n = d$. Thus (ii) holds. □

Let us look at the "true" metaplectic representation, where Φ is a quadratic map, and assume that $n = d$. By Assumption 1, X is asubset of regular points for Φ and Y is its image. Then, by Bezout's theorem, the number of points in $\Phi^{-1}(y_0)$ is at most 2^d. Thus, if U is reproducing, then by (i) in Corollary 15 the stabilizers are almost all compact. This is one of the reasons for studying the case of compact stabilizers in some detail.

3.7. **Compact stabilizers.** As a preliminary step, we assume that for a given $\bar{y} \in \overline{Y}$ the stabilizer $H_{\bar{y}}$ is compact. Clearly, the compactness of the stabilizer is independent of the choice of the origin $y_0 \in \pi^{-1}(\bar{y})$.
The compactness of the stabilizer makes available Schur's orthogonality relations for computing the inner integral over $H_{\bar{y}}$ in (32). Indeed, since $H_{\bar{y}}$ is compact, the representation $\Lambda_{\bar{y}}$ is completely reducible. Hence, for each equivalence class $\hat{s}$ in the dual group $\widehat{H}_{\bar{y}}$, we can choose a closed subspace $\mathcal{K}_{\bar{y},\hat{s}} \subset \mathcal{K}_{\bar{y}}$ such that the restriction $\Lambda_{\bar{y},\hat{s}}$ of $\Lambda_{\bar{y}}$ to $\mathcal{K}_{\bar{y},\hat{s}}$ belongs to $\hat{s}$, and we denote by $m_{\hat{s}}$ the multiplicity of $\hat{s}$ in $\Lambda_{\bar{y}}$ (with the convention that $\mathcal{K}_{\bar{y},\hat{s}} = 0$ if $m_{\hat{s}} = 0$). The following direct decomposition in primary inequivalent representations holds true

$$
\mathcal{K}_{\bar{y}} \simeq \bigoplus_{\hat{s} \in \widehat{H}_{\bar{y}}} \mathcal{K}_{\bar{y},\hat{s}} \otimes \mathbb{C}^{m_{\hat{s}}} \qquad \Lambda_{\bar{y}} \simeq \bigoplus_{\hat{s} \in \widehat{H}_{\bar{y}}} \Lambda_{\bar{y},\hat{s}} \otimes \mathrm{id},
\tag{36}
$$

where we interpret $\mathbb{C}^{m_{\hat{s}}} = \ell^2$ whenever $m_{\hat{s}} = \infty$. Mackey's theorem on induced representations of semi-direct products [22] guarantees that each induced representation $\mathrm{Ind}^G_{\mathbb{R}^d \rtimes H_{\bar{y}}}(e^{-2\pi i \langle y_0, \cdot \rangle} \Lambda_{\bar{y},\hat{s}})$ is irreducible on $\mathcal{H}_{\bar{y},\hat{s}}$ and gives the following direct decomposition in primary inequivalent representations for $W_{\bar{y}}$:

$$
\mathcal{H}_{\bar{y}} \simeq \bigoplus_{\hat{s} \in \widehat{H}_{\bar{y}}} \mathcal{H}_{\bar{y},\hat{s}} \otimes \mathbb{C}^{m_{\hat{s}}} \qquad W_{\bar{y}} \simeq \bigoplus_{\hat{s} \in \widehat{H}_{\bar{y}}} \mathrm{Ind}^G_{\mathbb{R}^d \rtimes H_{\bar{y}}}(e^{-2\pi i \langle y_0, \cdot \rangle} \Lambda_{\bar{y},\hat{s}}) \otimes \mathrm{id}\,.
\tag{37}
$$

Choose a basis $\{e_i\}$ of $\mathbb{C}^{m_{\hat{s}}}$ and, according to (36) and (37) respectively, write

$$F_{\bar{y}} = \sum_{\hat{s}\in\widehat{H}_{\bar{y}}} \sum_{i=1}^{m_{\hat{s}}} F_{\bar{y},\hat{s},i} \otimes e_i, \qquad F_{\bar{y}} \in \mathcal{H}_{\bar{y}}$$

$$u = \sum_{\hat{s}\in\widehat{H}_{\bar{y}}} \sum_{i=1}^{m_{\hat{s}}} u_{\hat{s},i} \otimes e_i, \qquad u \in \mathcal{K}_{\bar{y}}.$$

Also, we write $\operatorname{vol} H_{\bar{y}}$ for the mass of the compact group $H_{\bar{y}}$ relative to the unique Haar measure ds that makes formula (30) work. Note that $\operatorname{vol} H_{\bar{y}}$ is not necessarily one.

PROPOSITION 16. *Let $\bar{y} \in \overline{Y}$ be such that the stabilizer $H_{\bar{y}}$ is compact. Given $F_{\bar{y}} \in \mathcal{H}_{\bar{y}}$ the following facts are equivalent:*

(i) equality (32) holds true for all $u \in \mathcal{K}_{\bar{y}}$;
(ii) for all $\hat{s} \in \widehat{H}_{\bar{y}}$ such that $m_{\hat{s}} \neq 0$, and for all $i, j = 1, \dots, m_{\hat{s}}$

$$\int_Y \langle F_{\bar{y},\hat{s},i}(h(y)), F_{\bar{y},\hat{s},j}(h(y))\rangle_{\mathcal{K}_{\bar{y},\hat{s}}} \frac{\alpha(h(y))}{\Delta_G(h(y))} d\tau_{\bar{y}}(y) = \frac{\dim \mathcal{K}_{\bar{y},\hat{s}}}{\operatorname{vol} H_{\bar{y}}} \delta_{ij}. \tag{38}$$

Proof. Take $u \in \mathcal{K}_{\bar{y}}$. We compute the inner integral in (32) using Schur's orthogonality relations. For $\tau_{\bar{y}}$-almost every $y \in Y$

$$\int_{H_{\bar{y}}} |\langle u, \Lambda_{\bar{y},s} F_{\bar{y}}(h(y))\rangle_{\mathcal{K}_{\bar{y}}}|^2 \, ds = \sum_{\hat{s}\in\widehat{H}_{\bar{y}}} \sum_{i,j=1}^{m_{\hat{s}}} \langle u_{\hat{s},i}, u_{\hat{s},j}\rangle_{\mathcal{K}_{\bar{y},\hat{s}}} \langle F_{\bar{y},\hat{s},j}(h(y)), F_{\bar{y},\hat{s},i}(h(y))\rangle_{\mathcal{K}_{\bar{y},\hat{s}}} \frac{\operatorname{vol} H_{\bar{y}}}{\dim \mathcal{K}_{\bar{y},\hat{s}}}.$$

Choosing $u = u_{\hat{s},i}$, (32) is equivalent to

$$\int_Y \|F_{\bar{y},\hat{s},i}(h(y))\|^2_{\mathcal{K}_{\bar{y},\hat{s}}} \frac{\alpha(h(y))}{\Delta_G(h(y))} d\tau_{\bar{y}}(y) = \frac{\dim \mathcal{K}_{\bar{y},\hat{s}}}{\operatorname{vol} H_{\bar{y}}}. \tag{39}$$

Choose next $j \neq i$ and $u = u_{\hat{s},i} \oplus u_{\hat{s},j}$. Taking (39) into account, (32) is equivalent to

$$\int_Y \langle F_{\bar{y},\hat{s},i}(h(y)), F_{\bar{y},\hat{s},j}(h(y))\rangle_{\mathcal{K}_{\bar{y},\hat{s}}} \frac{\alpha(h(y))}{\Delta_G(h(y))} d\tau_{\bar{y}}(y) = 0.$$

Hence (i) is equivalent to (ii). □

Equation (38) has the following easy interpretation in terms of the abstract theory developed by Fuhr [14]. Indeed, for each irreducible representation in the decomposition (37), we can define the (possibly unbounded) operator $d_{\bar{y},\hat{s}}$ on $\mathcal{H}_{\bar{y},\hat{s}}$

$$d_{\bar{y},\hat{s}} F_{\bar{y},\hat{s}}(h) = \frac{\dim \mathcal{K}_{\bar{y},\hat{s}}}{\operatorname{vol} H_{\bar{y}}} \Delta_G(h) \, F_{\bar{y},\hat{s}}(h),$$

which satisfies (K2) precisely because the stabilizer is compact. The operator $d_{\bar{y},\hat{s}}$ is a positive self-adjoint injective operator semi-invariant with weight Δ_G^{-1} [10]. Now, (38) says that $F_{\bar{y},\hat{s},i}$ is in the domain of $d_{\bar{y},\hat{s}}^{-1/2}$ and

$$\langle d_{\bar{y},\hat{s}}^{-1/2} F_{\bar{y},\hat{s},i}, d_{\bar{y},\hat{s}}^{-1/2} F_{\bar{y},\hat{s},j}\rangle_{\mathcal{H}_{\bar{y},\hat{s}}} = \delta_{ij}, \qquad i, j = 1, \dots, m_{\hat{s}}. \tag{40}$$

One should compare this with Theorem 4.20 and equations (4.15) and (4.16) of [14].

As a consequence of the above discussion, we have the following easy characterization of existence.

Corollary 17. *Let $\bar{y} \in \overline{Y}$ be such that the stabilizer $H_{\bar{y}}$ is compact. There exists $F_{\bar{y}}$ such that equality* (32) *holds true for all $u \in \mathcal{K}_{\bar{y}}$ if and only if $m_{\hat{s}} \leq \dim(\mathcal{H}_{\bar{y},\hat{s}})$ for all $\hat{s} \in \widehat{H_{\bar{y}}}$. If G is non-unimodular, this last condition is always satisfied.*

Proof. Fix $\hat{s} \in \widehat{H_{\bar{y}}}$ such that $m_{\hat{s}} \neq 0$. If G is unimodular, $d_{\bar{y},\hat{s}}$ is the identity up to a multiplicative constant, so that $\{F_{\bar{y},\hat{s},i}\}_{i=1}^{m_{\hat{s}}}$ satisfying (40) are precisely the orthogonal families in $\mathcal{H}_{\bar{y},\hat{s}}$ with square norm equal to $\dim \mathcal{K}_{\bar{y},\hat{s}} / \operatorname{vol} H_{\bar{y}}$, whose existence is equivalent to $m_{\hat{s}} \leq \dim(\mathcal{H}_{\bar{y},\hat{s}})$.
If G is non-unimodular, since $d_{\bar{y},\hat{s}}$ is a semi-invariant operator with weight Δ_G, then its spectrum is unbounded (see (2) of [10]), so that $\dim \mathcal{H}_{\bar{y},\hat{s}} = +\infty$, provided that $m_{\hat{s}} \neq 0$. Hence the families $\{F_{\bar{y},\hat{s},i}\}_{i=1}^{m_{\hat{s}}}$ satisfying (40) are the families in the domain of $d_{\bar{y},\hat{s}}^{-1/2}$ that are orthonormal with respect to the inner product induced by $d_{\bar{y},\hat{s}}^{-1/2}$. □

If G is unimodular, (ii) of Corollary 15 implies that $\mathcal{K}_{\bar{y}}$ is finite-dimensional, so that $m_{\hat{s}} = 0$ for all but finitely many $\hat{s} \in \widehat{H}_{\bar{y}}$ for which $m_{\hat{s}}$ is finite. Furthermore, often the orbit $\pi^{-1}(\bar{y})$ is not finite, so that $\dim \mathcal{H}_{\bar{y},\hat{s}} = +\infty$ and the requirement $m_{\hat{s}} \leq \dim \mathcal{H}_{\bar{y},\hat{s}}$ is trivially satisfied for every $\hat{s} \in \widehat{H}_{\bar{y}}$.

From now on we assume that almost every stabilizer $H_{\bar{y}}$ is compact. For each $\bar{y}$ we can apply Proposition 16. The only non-trivial point is measurability. The following theorem addresses this problem by providing an explicit decomposition of the representation W, hence of U, as a direct integral of its irreducible components, each of which is realized as induced representation of the restriction of $\Lambda_{\bar{y}}$ to a suitable (irreducible) subspace. The result does not depend on the fact that U is reproducing. To state the theorem, we fix a Borel section $o : \overline{Y} \to Y$ whose existence is ensured by Assumption 2 [11], thereby choosing $o(\bar{y})$ as the origin of the orbit $\pi^{-1}(\bar{y})$. For all $\bar{y} \in \overline{Y}$, we set $\mathcal{K}_{\bar{y}} = L^2(X, \nu_{o(\bar{y})})$ and we regard $\bar{y} \mapsto K_{\bar{y}}$ as a λ-measurable field of Hilbert spaces with respect to the measurable structure induced by $C_c(X) \subset K_{\bar{y}}$.

Theorem 18. *Assume that the stabilizers $H_{\bar{y}}$ are compact for λ-almost every $\bar{y} \in \overline{Y}$. There exist a countable family $\{\bar{y} \mapsto \mathcal{K}_{\bar{y}}^n\}_{n \in \mathcal{N}}$ of measurable fields of Hilbert subspaces, $\mathcal{K}_{\bar{y}}^n \subset \mathcal{K}_{\bar{y}}$, and a family of cardinals $\{m_n\}_{n \in \mathcal{N}} \subset \{1, \ldots, \aleph_0\}$ such that, for almost every $\bar{y} \in \overline{Y}$,*

$$\mathcal{K}_{\bar{y}} \simeq \bigoplus_{n \in \mathcal{N}} \mathcal{K}_{\bar{y}}^n \otimes \mathbb{C}^{m_n}, \tag{41}$$

$$\Lambda_{\bar{y}} \simeq \bigoplus_{n \in \mathcal{N}} \Lambda_{\bar{y}}^n \otimes \mathrm{id}, \tag{42}$$

where (42) *is the decomposition of $\Lambda_{\bar{y}}$ into irreducibles.*

Before the proof, we add some remarks.

Remark 1. For each $n \in \mathcal{N}$ and for almost every $\bar{y} \in \overline{Y}$ we denote by $\mathcal{H}_{\bar{y}}^n$ the Hilbert space carrying the induced representation $\operatorname{Ind}_{\mathbb{R}^d \rtimes H_{\bar{y}}}(e^{-2\pi i \langle o(\bar{y}), \cdot \rangle} \Lambda_{\bar{y}}^n)$. Theorem 10.1 of

[22] ensures that, for each $n \in \mathcal{N}$, $\bar{y} \mapsto \mathcal{H}^n_{\bar{y}}$ is a measurable field of Hilbert subspaces, $\mathcal{H}^n_{\bar{y}} \subset \mathcal{H}_{\bar{y}}\}$ and

$$\mathcal{H} \simeq \bigoplus_{n\in\mathcal{N}} \int_{\overline{Y}} \mathcal{H}^n_{\bar{y}}\, d\lambda(\bar{y}) \otimes \mathbb{C}^{m_n} \tag{43}$$

$$W \simeq \bigoplus_{n\in\mathcal{N}} \int_{\overline{Y}} \mathrm{Ind}_{\mathbb{R}^d \rtimes H_{\bar{y}}}(e^{-2\pi i\langle o(\bar{y}),\cdot\rangle}\, \Lambda^n_{\bar{y}})\, d\lambda(\bar{y}) \otimes \mathrm{id} \tag{44}$$

where, by Theorem 14.1 of [22] each component $\mathrm{Ind}_{\mathbb{R}^d \rtimes H_{\bar{y}}}(e^{-2\pi i\langle o(\bar{y}),\cdot\rangle}\, \Lambda^n_{\bar{y}})$ is irreducible and two of them are inequivalent.

Remark 2. In the statement of the above theorem, given $n \in \mathcal{N}$, it is possible that for some $\bar{y} \in \overline{Y}$ the Hilbert space $\mathcal{K}^n_{\bar{y}}$ reduces to zero as well as $\mathcal{H}^n_{\bar{y}}$. If it is the case, then clearly $\Lambda^n_{\bar{y}}$ and $\mathrm{Ind}_{\mathbb{R}^d \rtimes H_{\bar{y}}}(e^{-2\pi i\langle o(\bar{y}),\cdot\rangle}\, \Lambda^n_{\bar{y}})$ can be removed from the corresponding integral decompositions of $\Lambda_{\bar{y}}$ and W.

Remark 3. Fix $\bar{y}$ and compare (36) with (42). The set $\mathcal{N}$ is a parametrization of the relevant elements in the dual group $\widehat{H_{\bar{y}}}$ defined by the direct decomposition of $\Lambda_{\bar{y}}$ into its irreducible components $\Lambda^n_{\bar{y}}$. In other words, for each $n \in \mathcal{N}$ for which $\mathcal{K}^n_{\bar{y}} \neq 0$ there exists $\hat{s}_n \in \widehat{H_{\bar{y}}}$ such that $\Lambda^n_{\bar{y}} = \Lambda_{\bar{y},\hat{s}_n}$ and $m_n = m_{\hat{s}}$ is its multiplicity, which is independent of $\bar{y}$ by its very construction.

Remark 4. As a consequence of the above theorem and general results about direct integrals, for each $n \in \mathcal{N}$ there exists a measurable field $\{\bar{y} \mapsto \varepsilon^n_{\bar{y},\ell}\}_{\ell\geq 1}$ of Hilbert bases for each field $\bar{y} \mapsto \mathcal{H}^n_{\bar{y}}$. Denoting by $\{e_j\}_{j=1}^{m_n}$ the canonical basis of $\mathbb{C}^{m_n}$, for any $F \in \mathcal{H}$

$$\begin{aligned} F &= \sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \int_{\overline{Y}} F^n_{\bar{y},j}\, d\lambda(\bar{y}) \otimes e_j \\ F^n_{\bar{y},j} &= \sum_{\ell\geq 1} f^n_{j,\ell}(\bar{y})\, \varepsilon^n_{\bar{y},\ell} \end{aligned} \tag{45}$$

where $\bar{y} \mapsto f^n_{j,\ell}(\bar{y})$ is a measurable complex function and

$$\|F\|^2 = \sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \int_{\overline{Y}} \|F^n_{\bar{y},j}\|^2_{\mathcal{H}^n_{\bar{y}}} d\lambda(\bar{y}) = \sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \sum_{\ell\geq 1} \int_{\overline{Y}} |f^n_{j,\ell}(\bar{y})|^2 d\lambda(\bar{y}). \tag{46}$$

Conversely, if $\{\bar{y} \mapsto f^n_{j,\ell}(\bar{y})\}_{n,j,\ell}$ is a family of measurable complex functions such that

$$\sum_{n\in\mathcal{N}} \int_{\overline{Y}} \sum_{j=1}^{m_n} \sum_{\ell\geq 1} |f^n_{j,\ell}(\bar{y})|^2 d\lambda(\bar{y}) < +\infty,$$

then (45) defines an element $F \in \mathcal{H}$.

Proof of Theorem 18. The proof is divided in several steps.

Step 1. We claim that there exists a sequence of Borel measurable functions $\xi_k : Y \to H$ such that, for any $y \in Y$, the set $\{\xi_k(y)\}_{k\in\mathbb{N}}$ is dense in the stability subgroup H_y. Define the map $\Xi : Y \times H \to Y \times Y$, $\Xi(y,h) = (h[y], y)$, which is continuous and, hence, Borel measurable. Furthermore, the diagonal $D = \{(y,y) \mid y \in Y\}$ is a Borel subset and, for any $y \in Y$, $H_y = \{h \in H \mid \Xi(y,h) \in D\}$. The Aumann's measurable selection principle, see for example Theorem III.23 of [5], ensures there exists a sequence of measurable functions $\xi_k : Y \to H$ such that the set $\{\xi_k(y)\}_{k\in\mathbb{N}}$ is dense in H_y for all $y \in Y$.

Step 2. For all $\bar{y} \in \overline{Y}$, set $M_{\bar{y}}$ equal to the von Neumann algebra on $\mathcal{K}_{\bar{y}}$ generated by the representation $\Lambda_{o(\bar{y})}$ of $H_{o(\bar{y})}$. We show that $\bar{y} \mapsto M_{\bar{y}}$ is a measurable field of von Neumann algebras. For each $\bar{y} \in \overline{Y}$ the result of Step 1 and the fact that $s \mapsto \Lambda_{\bar{y},s}$ is continuous implies that the sequence $\{\Lambda_{\bar{y},\xi_k(o(\bar{y}))}\}_{k\in\mathbb{N}}$ generates $M_{\bar{y}}$. Hence, it is enough to prove that for any $k \in \mathbb{N}$ the fields of operators $\bar{y} \mapsto \Lambda_{\bar{y},\xi_k(o(\bar{y}))}$ is λ-measurable by observing that for all $\varphi, \varphi' \in C_c(X)$ the map

$$y \to \langle \Lambda_{y,\xi_k(y)}\varphi, \varphi' \rangle_{\nu_y} = \int_X \sqrt{\alpha(\xi_k(y)^{-1})\beta(\xi_k(y)^{-1})}\varphi(\xi_k(y)^{-1}.x)\overline{\varphi'(x)}d\nu_y(x)$$

is Lebesgue measurable since ξ_k is Borel measurable, the map

$$h \to \int_X \sqrt{\alpha(h^{-1})\beta(h^{-1})}\varphi(h^{-1}.x)\overline{\varphi'(x)}d\nu_y(x)$$

is continuous, and the family $\{\nu_y\}$ is Lebesgue scalarly-integrable. Hence the corresponding decomposable von Neumann algebra $M = \int_{\overline{Y}} M_{\bar{y}}\, d\lambda(\bar{y})$ acts on the separable Hilbert space $\mathcal{K} = \int_{\overline{Y}} \mathcal{K}_{\bar{y}}\, d\lambda(\bar{y})$, see Prop. 1, Ch II § 3 of [8]. Since $\overline{Y}$ is second countable, Theorem 4, Ch II § 3 of [8] ensures that

$$M' = \int_{\overline{Y}} {M_{\bar{y}}}'\, d\lambda(\bar{y})$$
$$M \cap M' = \int_{\overline{Y}} M_{\bar{y}} \cap {M_{\bar{y}}}'\, d\lambda(\bar{y})$$

Step 3. For almost every $\bar{y} \in \overline{Y}$, $H_{\bar{y}}$ is a group of type I, hence $\Lambda_{\bar{y}}$ is of type I representation, that is, by definition, $M_{\bar{y}}$ is a type I von Neumann algebra. Since $\overline{Y}$ is second countable, result (A50) of [9] implies that M is of type I, as well as, M' by Theorem 1 Ch. 1 § 8 of [8]. Result (A50) of [9] applied to M' ensures the existence of a countable family $\{P^i\}_{i\in I}$ of non-zero projections in $M \cap M'$ with sum 1 such that each reduced algebra M'_{P^i} is of type I_{n_i} where $n_i \neq n_j$ if $i \neq j$. Hence, by definition, for each $i \in I$, there exists a family $\{P^{ij}\}_{j=1}^{n_i}$ of equivalent abelian pairwise orthogonal projections in M' with sum P^i. Since M' is decompasable

$$P^i = \int_{\overline{Y}} P^i_{\bar{y}}\, d\lambda(\bar{y})$$
$$P^{ij} = \int_{\overline{Y}} P^{ij}_{\bar{y}}\, d\lambda(\bar{y})$$

where, for almost every $\bar{y} \in \overline{Y}$,

i) $(P^i_{\bar{y}})_{i\in I}$ is a family of non-zero projections in $M_{\bar{y}} \cap M'_{\bar{y}}$ with sum 1;

ii) for each $i \in I$, $(P^{ij}_{\bar{y}})_{j=1}^{n_i}$ is a family of equivalent abelian pairwise orthogonal projections in $\cap M'_{\bar{y}}$ with sum $P^i_{\bar{y}}$.

By definition of equivalent projections, for any $i \in I$ and $1 < j \leq n_i$, there exists $U^{ij} \in M'$ such that $P_{i1} = (U^{ij})^* U^{ij}$ and $P^{ij} = U^{ij}(U^{ij})^*$, hence,

$$U^{ij} = \int_{\overline{Y}} U^{ij}_{\bar{y}} \, d\lambda(\bar{y})$$

where, for almost every $\bar{y} \in \overline{Y}$, $U^{ij}_{\bar{y}} \in M_{\bar{y}}{}'$ and $P^{i1}_{\bar{y}} = (U^{ij}_{\bar{y}})^* U^{ij}_{\bar{y}}$ and $P^{ij}_{\bar{y}} = U^{ij}_{\bar{y}}(U^{ij}_{\bar{y}})^*$. Define

$$\mathcal{K}^i = P^{i1}\mathcal{K} = \int_{\overline{Y}} \mathcal{K}^i_{\bar{y}} \, d\lambda(\bar{y}) \qquad \mathcal{K}^i_{\bar{y}} = P^{i1}_{\bar{y}} \mathcal{K}_{\bar{y}},$$

where, for each $i \in i$, $\bar{y} \mapsto \mathcal{K}^i_{\bar{y}}$ is a measurable field of Hilbert spaces by Proposition 9 Ch. II, § 1.7 of [8]. Hence, up to a unitary equivalence,

$$\mathcal{K} = \bigoplus_{i\in I} \mathcal{K}^i \otimes \mathbb{C}^{n_i} \qquad \mathcal{K}^i = \int_{\overline{Y}} \mathcal{K}^i_{\bar{y}} \, d\lambda(\bar{y}) \qquad \mathcal{K}_{\bar{y}} = \bigoplus_{i\in I} \mathcal{K}^i_{\bar{y}} \otimes \mathbb{C}^{n_i}$$

and, for almost every $\bar{y} \in \overline{Y}$ and each $i \in I$

i) M leaves invariant $\mathcal{K}^i$ and, denoted by M^i the reduced algebra, $(M^i)'$ is abelian;
ii) $M_{\bar{y}}$ leaves invariant $\mathcal{K}^i_{\bar{y}}$ and, denoted by $M_{\bar{y}}{}^i$ the reduced algebra, $(M^i_{\bar{y}})'$ is abelian
iii) $\bar{y} \mapsto M^i_{\bar{y}}$ is a measurable field of von Neumann algebras and

$$M^i = \int_{\overline{Y}} M^i_{\bar{y}} \, d\lambda(\bar{y})$$

see Proposition 6 Ch II § 3.5 of [8].
iv) by Proposition 3 Ch II § 3.4 of [8]

$$M = \Pi_{i\in I} M^i \otimes \mathbb{C}\,\mathrm{id}_{n_i} \qquad M_{\bar{y}} = \Pi_{i\in I} M^i_{\bar{y}} \otimes \mathbb{C}\,\mathrm{id}_{n_i}$$

where $\mathbb{C}\,\mathrm{id}_{n_i}$ denotes the trivial algebra acting on an n_i-dimensional Hilbert space.

Step 4. We fix $i \in I$. Result (A50) of [9] applied to M^i ensures the existence of a countable family $\{Q^{it}\}_{t\in T_i}$ of non-zero projections in $M^i \cap (M^i)'$ with sum 1 such that each reduced algebra $M_{Q^{it}}$ is of type $I_{m_{it}}$ where $m_{it} \neq m_{it'}$ if $t \neq t'$. Hence, reasoning as above with self-explained notation, up to a unitary equivalence,

$$\mathcal{K}^i = \bigoplus_{t\in T_i} \mathbb{C}^{m_{it}} \otimes \mathcal{T}^{it} \qquad \mathcal{T}^{it} = \int_{\overline{Y}} \mathcal{T}^{it}_{\bar{y}} \, d\lambda(\bar{y}) \qquad \mathcal{K}^i_{\bar{y}} = \bigoplus_{t\in T_i} \mathbb{C}^{m_{it}} \otimes \mathcal{T}^{it}_{\bar{y}}$$

where $\mathbb{C}^{m_{it}}$ is a (fixed) Hilbert space of dimension m_{it} and $\bar{y} \mapsto \mathcal{T}^{it}_{\bar{y}}$ is a measurable filed of Hilbert spaces. For almost every $\bar{y} \in \overline{Y}$ and each $t \in T^i$

i) M^i leaves invariant $\mathcal{K}^{it}$ and, denoted by M^{it} the reduced algebra,

$$(M^{it})' = \mathbb{C}\,\mathrm{id}_{m_{it}} \otimes \mathcal{A}^{it} \qquad (\mathcal{A}^{it})' = \mathcal{A}^{it}$$

where $\mathcal{A}^{it}$ acts on $\mathcal{T}^{it}$.
ii) $M^{it}_{\bar{y}}$ leaves invariant $\mathcal{K}^{it}_{\bar{y}}$ and, denoted by $M^{it}_{\bar{y}}$ the reduced algebra,

$$(M^i_{\bar{y}})' = \mathbb{C}\mathrm{id}_{m_{it}} \otimes \mathcal{A}^{it}_{\bar{y}} \qquad (\mathcal{A}^{it}_{\bar{y}})' = \mathcal{A}^{it}_{\bar{y}}$$

where $\mathcal{A}^{it}_{\bar{y}}$ acts on $\mathcal{T}^{it}_{\bar{y}}$.

iii) $\bar{y} \mapsto \mathcal{A}^{it}_{\bar{y}}$ is a measurable field of von Neumann algebras and

$$\mathcal{A}^{it} = \int_{\overline{Y}} \mathcal{A}^{it}_{\bar{y}} \, d\lambda(\bar{y})$$

Furthermore

$$M^i = \Pi_{t\in T_i} \, \mathcal{L}(\mathbb{C}^{m_{it}}) \otimes \mathcal{A}^{it} \qquad M^i_{\bar{y}} = \Pi_{t\in T_i} \, \mathcal{L}(\mathbb{C}^{m_{it}}) \otimes \mathcal{A}^{it}_{\bar{y}}$$

Step 5. Fix $i \in I$ and $t \in T_i$, for almost every $\bar{y} \in \overline{Y}$ the definition of $M_{\bar{y}}$ and the fact that $H_{\bar{y}}$ is a compact group imply that the representation $\Lambda_{\bar{y}}$ leaves invariant each copy $\mathbb{C}^{m_{it}}$, the corresponding restriction is irreducible, and two different restrictions are inequivalent. This means that $\mathcal{A}^{it}_{\bar{y}}$ is spatially isomorphic to the diagonal algebra acting on $\ell_2(\widehat{S}_{it})$ where $\widehat{S}_{it}$ is the subset of $\widehat{H_{\bar{y}}}$ defined by the irreducible components of $\Lambda_{\bar{y}}$, restricted to $\mathcal{K}^i_{\bar{y}}$, acting on $\mathbb{C}^{m_{it}}$. In particular, $\dim \mathcal{T}^{it}_{\bar{y}} = \operatorname{card} \widehat{S}_{it}$.

Step 6. Fixed $i \in I$ and $t \in T_i$ for any $k \in \{1, \dots, \aleph_0\}$

$$\overline{Y}_{itk} = \{\bar{y} \in \overline{Y} \mid \dim \mathcal{T}^{it}_{\bar{y}} = k\}$$

is a Borel subset of $\overline{Y}$ by Proposition 1 Ch. II, § 1 of [8]. and, for almost every $\bar{y} \in \overline{Y}_{itk}$, $M_{\bar{y}it}$ is spatially isomorphic to $\mathcal{L}(\mathbb{C}^{m_{it}}) \otimes \ell_\infty(I_k)$, where

$$I_k = \begin{cases} \{1, \dots, k\} & k < +\infty \\ \mathbb{N} & k = +\infty \end{cases}.$$

Lemma 2 Ch. 2 § 3 of [8] implies that the isomorphism can be realized by a unitary operator $W^{it}_{\bar{y}} : \ell_2(I_k) \to \mathcal{T}^{ik}_{\bar{y}}$ such that $\bar{y} \mapsto W^{it}_{\bar{y}}$ is a measurable field of operators. If $\{f_\ell\}_{\ell \in I_k}$ denotes the canonical base of $\ell_2(I_k)$, define for each $\ell \in \mathbb{N}$

$$\begin{cases} \mathcal{K}^n_{\bar{y}} := \mathcal{K}^{i,t,\ell}_{\bar{y}} = \mathbb{C}^{m_{it}} \otimes W^{it}_{\bar{y}} \, f_\ell & \bar{y} \in \overline{Y}_{itk} \text{ and } \ell \in I_k \\ \mathcal{K}^n_{\bar{y}} := \mathcal{K}^{i,t,\ell}_{\bar{y}} = 0 & \text{otherwise} \end{cases}$$

where we label the triple of indexes (i, t, ℓ) simply by a unique label n, running over a countable set $\mathcal{N}$. By construction, the following properties hold true

a) for each $n \in \mathcal{N}$, $\bar{y} \mapsto \mathcal{K}^n_{\bar{y}}$ is a measurable field of Hilbert subspaces, $\mathcal{K}^n_{\bar{y}} \subset \mathcal{K}_{\bar{y}}$, and

$$\mathcal{K}_{\bar{y}} = \bigoplus_{n\in\mathcal{N}} \mathcal{K}^n_{\bar{y}} \otimes \mathbb{C}^{m_n}$$

where $m_n = n_i$ if $n = \{i, t, \ell\}$.

b) for almost every $\bar{y} \in \overline{Y}$, the representation $\Lambda_{\bar{y}}$ leaves invariant each $\mathcal{K}^n_{\bar{y}}$, the corresponding restriction $\Lambda_{\bar{y},n}$ is an irreducible representation of $H_{\bar{y}}$, these representations are pairwise inequivalent and

$$\Lambda_{\bar{y}} = \bigoplus_{n\in\mathcal{N}} \Lambda_{\bar{y},n} \otimes \mathrm{id}$$

□

By means of the intertwining operator S given by Theorem 13, the direct decomposition (44) gives rise to a corresponding direct decomposition of the mock-metaplectic representation U. Hence, the abstract theory of [14] applies to characterize the admissible vectors for U. However, we can direct apply Corollary 16.

We need a last technical lemma about the measurable property of the map $\bar{y} \mapsto \operatorname{vol}(H_{\bar{y}})$, compare with Lemma 18 of [15].

LEMMA 19. *Assume that for almost every $y_0 \in Y$ the stabilizer H_{y_0} is compact and define*

$$\operatorname{vol}(H_{y_0}) = \int_{H_{y_0}} ds$$

where ds is the unique Haar measure of H_{y_0} such that

$$\int_H \varphi(h)\alpha(h^{-1})dh = \int_Y \left(\int_{H_{y_0}} \varphi(h(y)s)ds\right) d\tau_{\pi(y_0)}(y) \qquad \varphi \in C_c(Y).$$

The following facts holds true:

(i) For all y_0 and $h \in H$, $\operatorname{vol}(H_{h[y_0]}) = \Delta_G(h^{-1}) \operatorname{vol}(H_{y_0})$.
(ii) The map $y_0 \mapsto \operatorname{vol}(H_{y_0})$ is Lebesgue measurable.

Furthermore, given a Borel measurable section $o : \overline{Y} \to Y$, the map

$$\bar{y} \mapsto \frac{\dim \mathcal{K}^n_{\bar{y}}}{\operatorname{vol}(H_{\bar{y}})}$$

is λ-measurable. If G is unimodular, the map is independent of the choice of the section o.

Proof. Fix a continuous function $f \in L^1(Y)$ such that $f(y) > 0$ for all $y \in Y$. The definition of $\tau_{\bar{y}}$ (see Theorem 8) with (iii) of Theorem 25 gives that for λ-almost every $\bar{y} \in Y$, f is $\tau_{\bar{y}}$-integrable. Clearly, the function $(y_0, h) \mapsto f(h[y_0])\alpha(h^{-1})$ is continuous on $Y \times H$. Given $y_0 \in Y$, let $\bar{y} = \pi(y_0)$ so that we can choose y_0 as the origin of $\pi^{-1}(\bar{y})$ and define ds as the unique Haar measure of $H_{y_0} = H_{\bar{y}}$ for which (30) holds true. By (ii) of Theorem 25 we have that, for almost all $y_0 \in Y$,

$$\begin{aligned}
0 < \int_H f(h[y_0])\alpha(h^{-1})\, dh &= \int_Y \left(\int_{H_{y_0}} \varphi(h_y s[y_0]), ds\right) d\tau_{\pi(y_0)}(y) \\
&= \operatorname{vol}(H_{y_0}) \int_Y f(y) d\tau_{\pi(y_0)}(y) < +\infty
\end{aligned}$$

since $h_y s[y_0] = y$, first inequality is due to the fact $f > 0$ and the last inequality follows from $f \in L^1(Y)$. Clearly $y_0 \mapsto \int_H f(h[y_0])\alpha(h^{-1})\, dh$ is Lebesgue-measurable as well as $y_0 \mapsto \int_Y f(y) d\tau_{\pi(y_0)}(y)0$ is Lebesgue measurable and strictly positive, so that $y_0 \mapsto \operatorname{vol}(H_{y_0})$ is λ-measurable, too. The fact that, for all $n \in \mathcal{N}$ the map $\bar{y} \mapsto \dim \mathcal{K}^n_{\bar{y}}$ is λ-measurable is consequence of Proposition 1, Ch. 2 § 1.4 of [8].
If $y_1 = \ell[y_0]$ for some $\ell \in H$, so that $\pi(y_0) = \pi(y_1)$, then as above

$$\begin{aligned}
\operatorname{vol}(H_{y_1}) \int_Y f(y) d\tau_{\pi(y_0)}(y) &= \int_H f(h[y_1])\alpha(h^{-1})\, dh \\
(\ h \mapsto h\ell^{-1}\ ) &= \Delta_H(\ell^{-1})\alpha(\ell) \int_H f(h[y_0])\alpha(h^{-1})\, dh \\
&= \Delta_G(\ell^{-1}) \operatorname{vol}(H_{y_0})
\end{aligned}$$

The second half of the lemma is clear. ∎

We are ready to state our main result about the characterization of the admissible vectors of G, we give separately according to G is unimodular or not.

We consider first the unimodular case, compare with Eq. (4.14) of Theorem 4.22 in [14].

THEOREM 20. *Assume that G is unimodular and for almost every $\bar{y} \in \overline{Y}$ the stabilizers $H_{\bar{y}}$ are compact. The representation U is reproducing if and only if the following two conditions hold true:*

a) fix any origin $y_0 \in \pi^{-1}(\bar{y})$, then

$$\int_{\overline{Y}} \frac{\operatorname{card} \Phi^{-1}(y_0)}{\operatorname{vol} H_{\bar{y}}} \, d\lambda(\bar{y}) < +\infty, \tag{47}$$

b) for all $n \in \mathcal{N}$ and for almost every $\bar{y} \in \overline{Y}$ such that $\mathcal{K}^n_{\bar{y}} \neq 0$

$$m_n \leq \dim \mathcal{K}^n_{\bar{y}} \tag{48}$$

with the notation in (41) and (42);

Under the above equivalent conditions, η is an admissible vector for U if and only if

$$S\eta = \sum_{n \in \mathcal{N}} \sum_{j=1}^{m_n} \int_{\overline{Y}} \sqrt{\frac{\dim \mathcal{K}_{\bar{y},n}}{\operatorname{vol} H_{\bar{y}}}} \varepsilon^n_{\bar{y},j} \, d\lambda(\bar{y}) \otimes e_j,$$

where $\{\bar{y} \mapsto \varepsilon^n_{\bar{y},\ell}\}_{j \geq 1}$ is any measurable field of Hilbert bases for $\bar{y} \mapsto \mathcal{H}^n_{\bar{y}}$ and $(e_i)_{i=1}^{m_n}$ is the canonical basis of $\mathbb{C}^{m_n}$.

Proof. With the notation as in Remark 4, Theorem 14 and Corollary 16 with $\Delta_G(h_y) = 1$ give that $\eta \in L^2(X)$ is an admissible vector for U if and only if $F = W\eta \in \mathcal{K}$ satisfies the following condition. Given $n \in \mathcal{N}$, for almost every $\bar{y} \in \overline{Y}$ such that $\mathcal{K}^n_{\bar{y}}$ (see Remark 2 and 3), for all $i, j = 1, \ldots, m_n$

$$\langle F^n_{\bar{y},i}, F^n_{\bar{y},j} \rangle_{\mathcal{H}^n_{\bar{y}}} = \delta_{i,j} \frac{\dim \mathcal{K}_{\bar{y},n}}{\operatorname{vol} H_{\bar{y}}},$$

that is, the family $\{F^n_{\bar{y},i}\}_{i=1}^{m_n}$ is orthogonal in $\mathcal{H}^n_{\bar{y}}$ and normalized with square norm $\dim \mathcal{K}_{\bar{y},n} / \operatorname{vol} H_{\bar{y}}$.
As a consequence, if η is an admissible vector, clearly (48) holds true and, with (46), we have that

$$\|F\|^2_{\mathcal{H}} = \int_{\overline{Y}} \left(\sum_{n \in \mathcal{N}} \sum_{i=1}^{m_n} \frac{\dim \mathcal{K}_{\bar{y},n}}{\operatorname{vol} H_{\bar{y}}} \right) d\lambda(\bar{y}) = \int_{\overline{Y}} \frac{\operatorname{card} \Phi^{-1}(y_0)}{\operatorname{vol} H_{\bar{y}}} \, d\lambda(\bar{y}),$$

and (47) follows. Conversely, define $F \in \mathcal{H}$ such that, for all $j = 1, \ldots, m_n$ and $\ell \geq 1$

$$f^n_{,j,\ell}(\bar{y}) = \delta_{j,\ell} \sqrt{\frac{\dim \mathcal{K}_{\bar{y},n}}{\operatorname{vol} H_{\bar{y}}}} \qquad \text{a.e.} \bar{y} \in \overline{Y},$$

which is possible due to (48). The functions $f_{n,j,\ell}$ are λ-measurable by Lemma 19. Eq. (47) ensures that (46) is finite (see the above string of equalities). ∎

We now consider the non-unimodular case. For all $n \in \mathcal{N}$ and for almost every $\bar{y} \in \overline{Y}$ we define the positive self-adjoint injective operator $d_{\bar{y},n}$ acts on $\mathcal{H}^n_{\bar{y}}$ by multiplication

$$(d_{\bar{y},n} F_{\bar{y},n})(h) = \frac{\dim \mathcal{K}_{\bar{y},n}}{\mathrm{vol}\, H_{\bar{y}}} \Delta_G(h)\, F_{\bar{y},n}(h) \qquad h \in H,$$

see Corollary 17 and its proof.

Theorem 21. *Assume that G is non-unimodular and that for almost every $\bar{y} \in \overline{Y}$ the stabilizer $H_{\bar{y}}$ is compact. Then U is reproducing and $\eta \in L^2(X)$ is an admissible vector for U if and only if $S\eta = \sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \int_{\overline{Y}} F^n_{\bar{y},j}\, d\lambda(\bar{y}) \otimes e_j$ is such that*

(i) for all $n \in \mathcal{N}$ and $i = 1, \ldots, m_n$, the map $\bar{y} \mapsto F_{\bar{y},n,i}$ is measurable field of vectors for $\{\mathcal{H}^n_{\bar{y}}\}$;

(ii) for any $n \in \mathcal{N}$ and for almost all $\bar{y} \in \overline{Y}$ such that $\mathcal{K}^n_{\bar{y}} \neq 0$

$$\langle d_{\bar{y},n}^{-1/2} F^n_{\bar{y},i}, d_{\bar{y},n}^{-1/2} F^n_{\bar{y},j} \rangle_{\mathcal{H}_{\bar{y},n}} = \delta_{ij} \qquad i,j = 1, \ldots, m_n$$

(iii) $\displaystyle\sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \int_{\overline{Y}} \| F^n_{\bar{y}.j} \|^2_{\mathcal{H}^n_{\bar{y}}} d\lambda(\bar{y}) < +\infty\,.$

Proof. The only non-trivial part is the existence of an admissible vector. This fact is a consequence of Theorem 4.23 of [14], whose proof can be repeated to our setting to provide a direct proof of the existence. We report only the idea.
Fix a strictly positive sequence such that $\sum_{n\in\mathcal{N}} \sum_{i=1}^{m_n} a_{n,i} < +\infty$. Since the stability subgroups are compact

$$(0, +\infty) = \Delta_G(H) = \Delta_G(H/H_{\bar{y}}) = \Delta_G(\{h(y) \mid y \in \pi^{-1}(\bar{y})\}).$$

and $y \mapsto \Delta_G(h(y))$ is continuous on $\pi^{-1}(\bar{y})$, for each $\bar{y} \in \overline{Y}$, there exists an subset $O_{\bar{y},n,i}$ of $\pi^{-1}(\bar{y})$ with strictly positive $\tau_{\bar{y}}$-measure such that for all $y \in O_{\bar{y},n,i}$,

$$\sup_{y \in Y_{\bar{y},n,i}} \frac{\dim \mathcal{K}_{\bar{y},n} \Delta_G(h_y)}{\mathrm{vol}\, H_{\bar{y}}} \leq a_{n,i}.$$

Select a family of measurable fields $\{\bar{y} \mapsto F^n_{\bar{y},j}\}_{j=1}^{m_n}$ of vectors in $\mathrm{dom}\, d_{\bar{y},n}^{-1/2}$, that are orthonormal with respect to the scalar product induced by $d_{\bar{y},n}^{-1/2}$ with the property that the support with respect to $\tau_{\bar{y}}$ of the map $y \mapsto \| F^n_{\bar{y},j}(h_y) \|^2_{\mathcal{K}(\bar{y})_n}$ is contained in $O_{\bar{y},n,i}$, then the third condition is satisfied since

$$\| F^n_{\bar{y},j} \|^2_{\mathcal{H}^n_{\bar{y}}} \leq \sup_{y \in O_{\bar{y},n,i}} \frac{\dim \mathcal{K}_{\bar{y},n} \Delta_G(h_y)}{\mathrm{vol}\, H_{\bar{y}}} \leq a_{n,i}.$$

□

4. Examples

We now discuss some of the examples we introduced in Section 2.

4.1. **Example 1.** In this example, the map Φ is the identity so that the set of critical points reduces to the empty set and Assumption 1 is satisfied with the choice $X = Y = \mathbb{R}^d$ (recall that $n = d$) and $\alpha(h)\beta(h) = 1$ for all $h \in H$. Clearly, for all $y \in \mathbb{R}^d$, $\Phi^{-1}(y)$ is a singleton, the corresponding measure ν_y is the trivial, so that Theorem 7 states that $\eta \in L^2(X)$ is an admissible vector for U, for λ-almost $\bar{y} \in \mathbb{R}^d/H$

$$\int_H |\eta(h^{-1}[y_0])|^2\, dh = 1$$

where y_0 is a fixed origin in $\pi^{-1}(\bar{y})$ and λ is a pseudo-image measure of the Lebesgue measure, we fix in the following. Since the above equation holds true for any other point in $\in \pi^{-1}(\bar{y})$, it follows that η is a weak admissible vector in the sense of Definition 7 of [15]. Assumption 2 is nothing else that the fact that the semi-direct product $\mathbb{R}^d \rtimes H$ is regular. Theorem 6 of the cited paper proves that this regularity is essentially necessary to have weak admissible vectors, see comment at the end of Section 3.4. Under this assumption, Corollary 15 ensures that the stabilizers $H_{\bar{y}}$ are compact for almost every $\bar{y} \in \overline{Y}$. Hence the results of Section 3.7 holds true. Clearly, for almost every $\mathcal{K}(\bar{y}) = \mathbb{C}$, $\mathcal{N}$ is a singleton and $m_n = \dim(\mathcal{K}^n_{\bar{y}}) = 1$, so that U is always reproducing if G is non-unimodular, otherwise it is equivalent to the fact that $\int_{\overline{Y}} d\lambda(\bar{y})/\operatorname{vol} H_{\bar{y}}$ is finite, which is precisely the content of Theorem 19 of [15], see also Section 5 of [14]. The factor $\operatorname{vol} H_{\bar{y}}$ is due to a different normalization of the Haar measures on the stabilizers.

4.2. **Example 2.** In this example $n = 2$ and $d = 1$ so that U is not reproducing. This fact is well known since G has a non-compact center and U is irreducible.

4.3. **Example 3.** The groups of the form (10) with $n = d$ are reproducing if and only if G is non-unimodular and the critical points of Φ are negligible.

We need to a result, which is of some interest by itself. The idea goes back to [21].

PROPOSITION 22. *Assume that Φ is a homogeneous map of degree $p > 0$ and the action on $\mathbb{R}^d$ is linear. If U is a reproducing representation, then G is non-unimodular.*

The proof is based on the following lemma.

LEMMA 23. *Assume that Φ is a homogeneous map of degree p and the action on $\mathbb{R}^d$ is linear. If η is an admissible vector for U, for any $\delta \in \mathbb{R}_+$, $\sqrt{\delta^{np-d}}\eta^\delta$ is an admissible vector, too.*

Proof. Denoted by $q = np - d$, the assumption of Φ implies that for all $x \in X$, $a \in \mathbb{R}^n$ and $\delta \in \mathbb{R}_+$,

$$\langle \Phi(\delta x), \delta^{-p} a \rangle = \langle \Phi(x), a \rangle. \tag{49}$$

Now clearly $\sqrt{\delta^q}\eta^\delta \in L^2(X)$ and, for all $f \in L^2(X)$, the linearity of $x \mapsto h.x$ gives

$$\begin{aligned}
\int_G |\langle f, U_g \sqrt{\delta^q}\eta^\delta\rangle|^2\, dg &= \delta^q \int_H \int_A |\int_X f(x)\beta(h)^{-\frac{1}{2}} e^{-2\pi i\langle \Phi(x),a\rangle}\eta(h.(\delta^{-1}x)dx|^2 \frac{dadh}{\alpha(h)} \\
(\ x \mapsto \delta x,\ a \mapsto \delta^{-p}a, (49)\ ) &= \delta^{q+2d-np} \int_G |\langle f^{\delta^{-1}}, U_g\eta\rangle|^2 dg \\
(\ \text{reproducing formula}\ ) &= \delta^{q+2d-np} \int_X |f(\delta x)|^2 dx \\
(\ x \mapsto \delta^{-1}x\ ) &= \delta^{q+d-np}\ \|f\|^2 = \|f\|^2,
\end{aligned}$$

so that $\sqrt{\delta^q}\eta^\delta$ is an admissible vector for U. □

Proof of Proposition 22. By contradiction assume that G is unimodular. Fix $\delta \in \mathbb{R}_+$. Choose an admissible vector $\eta \in Ł^2(X)$.

$$\begin{aligned}
\int_X |\eta(x)|^2 dx &= \delta^{-d} \int_X |\eta^\delta(x)|^2 dx \\
(\ \text{reproducing formula for } \eta\ ) &= \delta^{-d} \int_H \int_A |\langle \eta^\delta, U_{ah}\eta\rangle|^2 \frac{dadh}{\alpha(h)} \\
(\ a \mapsto -a,\ h \mapsto h^{-1}\ ) &= \delta^{-d} \int_H \int_A |\langle U_{(h^\dagger[a],h)}\eta^\delta, \eta\rangle|^2 \Delta_H(h^{-1}) \frac{dadh}{\alpha(h^{-1})} \\
(\ a \mapsto (h^\dagger)^{-1}[a]\ ) &= \delta^{-d} \int_H \int_A |\langle U_{(a,h)}\eta^\delta, \eta\rangle|^2 \Delta_G(h^{-1}) \frac{dadh}{\alpha(h)} \\
(\ q = np - d\ ) &= \delta^{-q-d} \int_G |\langle \eta, U_{g)}\sqrt{\delta^q}\eta^\delta\rangle|^2 dg \\
(\ \text{reproducing formula for } \sqrt{\delta^q}\eta\ ) &= \delta^{-np} \int_X |\eta(x)|^2 dx = \delta^{-np}\|\eta\|^2
\end{aligned}$$

Since $\|\eta\| \neq 0$ and $np \neq 0$, this is a contradiction. □

Come back to our example. If G is reproducing, then the set of critical points of Φ is negligible by (ii) of Theorem 3. The above proposition with $n = d$ and $p = 2$ implies that G is non-unimodular. Conversely, assume that G is non-unimodular and the critical points of Φ are negligible. Since Φ is a family of d-polynomial functions in d-variables and the Jacobian is different from zero almost everywhere, a standard result of algebra implies that $\Phi^{-1}(y)$ is a finite set for almost every y, hence (i) of Corollary 15 implies that the stabilizers are compact, and Theorem 21 shows that U is reproducing, as well as a characterization of the admissible vectors. However, one can also directly apply Theorem 7, taking into account that $\Phi(y)$ is a finite set. Explicitly the following corollary holds true.

Corollary 24. *A function $\eta \in L^2(X)$ is an admissible vector for U if and only if for λ-almost every $\bar{y} \in \Phi(\mathbb{R}^d)/H$, there exists $y \in \pi^{-1}(\bar{y})$ such that, for all points $x_1, \dots x_{N_y} \in \Phi^{-1}(y)$*

$$\int_H \eta(h^{-1}.x_i)\overline{\eta(h^{-1}.x_j)} \frac{dh}{\alpha(h)\beta(h)} = (J\Phi)(x_i)\ \delta_{ij} \qquad i,j = 1,\dots N_y.$$

If the above equation is satisfied for a pair $x_i, x_j \in \Phi^{-1}(\bar{y})$, it holds true for any pair $s.x_i, s.x_{\bar{y},j} \in \Phi^{-1}(\bar{y})$ where $s \in H_{\bar{y}}$.

Proof. We apply Theorem 7. Given $\bar{y} \in \Phi(X)/H$ and $y \in \pi^{-1}(\bar{y})$ for which (21) holds true, formula (58) gives that $\nu_y = \sum_{i=1}^{M} \frac{\delta_{x_i}}{(J\Phi)(x_i)}$.
Reasoning as in the proof of Proposition 16, (21) is equivalent to

$$\int_H \eta(h^{-1}.x_i)\overline{\eta(h^{-1}.x_j)} \frac{dh}{\alpha(h)\beta(h)} = (J\Phi)(x_i)\delta_{ij} \qquad i,j = 1, \dots N_y.$$

The last claim is evident since H_{y_0} is compact so that for all $s \in H_{\bar{y}}$ $\alpha(s) = \beta(s) = 1$ and the equality

$$(J\Phi)(h.x) = (J\Phi)(x)\alpha(h)^{-1}\beta(h)^{-1} \qquad h \in H.$$

□

As an example, we apply the above corollary to the metaplectic representation restricted to the group $TDS(2)$, considered at the end of Example 3 Note that

$$(J\Phi)(x_1, x_2) = x_2^2 \qquad \alpha(\delta, \ell) = \delta^2 \qquad \beta(\delta, \ell) = \delta^{-1}.$$

We set $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \neq 0\}$, which is an H-invariant open set with full Lebesgue measure and $Y = \Phi(X) = \mathbb{R} \times \mathbb{R}_-$, which is a transitive free H-space. We choose as origin the point $y_0 = (0, -1/2)$ so that $\Phi^{-1}(y_0) = \{(0, \pm 1)\}$. Since for any $h = (\delta, \ell) \in H$

$$h^{-1}.(0, \pm 1) = \sqrt{\delta}(\mp \ell/2, \pm 1),$$

a function $\eta \in L^2(X)$ is an admissible vector if and only if

$$\int_{(0,+\infty)\times\mathbb{R}} |\eta(\mp\frac{\sqrt{\delta}\ell}{2}, \pm\sqrt{\delta})|^2 \, \frac{d\delta d\ell}{\delta^2} = 1$$

$$\int_{(0,+\infty)\times\mathbb{R}} \eta(-\frac{\sqrt{\delta}\ell}{2}, \sqrt{\delta})\overline{\eta}(\frac{\sqrt{\delta}\ell}{2}, -\sqrt{\delta}) \, \frac{d\delta d\ell}{\delta^2} = 0.$$

With the change of variables $x_1 = -\frac{\sqrt{\delta}\ell}{2}$ and $x_2 = \sqrt{\delta}$, whose Jacobian is $\frac{1}{4}$ the above equations become

$$\int_{\mathbb{R}\times\mathbb{R}_+} |\eta(\pm x_1, \pm x_2)|^2 \, \frac{dx_1 dx_2}{x_2^4} = \frac{1}{4}$$

$$\int_{\mathbb{R}\times\mathbb{R}_+} \eta(x_1, x_2)\overline{\eta}(-x_1, -x_2) \, \frac{dx_1 dx_2}{x_2^4} = 0,$$

which are precisely (5.17) and (5.18) of Theorem 5 in [7], see also [6]. Note that U is equivalent to two copies of the irreducible representation associated with the shearlets, which up to a Fourier conjugation is $\mathrm{Ind}_{\mathbb{R}^2\times\{(1,0)\}}^{TDS(2)}(\chi)$ where χ is the character of $\mathbb{R}^2$ $(a_1, a_2) \mapsto e^{\pi i a_2}$, [16].

4.4. **Example 4.** Assumption 1 is satisfied with the choice of $X = \mathbb{R}^2 \setminus \{0\}$ and $Y = \Phi(X) = (0, +\infty)$ since X is a H-invariant open set whose complement has zero Lebesgue measure. The group H acts freely on Y so that the orbit space Y/H reduces to a singleton and Assumption 2 holds true. We choose $y_0 = 1$ as the origin of the orbit so that the corresponding stabilizer is the compact group $\mathcal{T} = H_1$. Since G is non-unimodular, U is reproducing by Theorem 21. To characterize its admissible vectors, we note that the measure λ is trivial, so that in Theorem 8 the relatively invariant measure on Y is $\tau_1 = dy$. Furthermore, the map

$$\xi \mapsto (\cos\xi, \sin\xi)$$

is diffeomorphism of S^1 onto the Riemannian submanifold $\Phi^{-1}(1) = \{x_1^2 + x_2^2 = 1\}$. The Riemannian measure on S^1 is $d\xi$ so that, for all $f \in C_c(X)$

$$\int_X \varphi(x_1, x_2) d\nu_1(x_1, x_2) = \int_0^{2\pi} \varphi(\cos\xi, \sin\xi) \frac{d\xi}{2}.$$

By setting $h(y) = (\sqrt{y}, 0)$ so that $h(y)[1] = y$, (30) gives the Haar measure ds on $\mathcal{T}$ is $\frac{d\theta}{4\pi}$ since

$$\int_H \varphi(t, \theta) t dt \frac{d\theta}{2\pi} = \int_0^{+\infty} \left(\int_0^{2\pi} \varphi(\sqrt{y}, \theta) \frac{d\theta}{4\pi} \right) dy,$$

so that $\operatorname{vol} H_1 = \frac{1}{2}$.
The representation Λ_1 of $\mathcal{T}$ on $L^2(X, \nu_1) \simeq L^2(S^1, d\xi/2)$ is the regular representation, so that

$$L^2(X, \nu_1) \simeq \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\,\{e^{in\xi}\}$$
$$\Lambda_{1,\theta} \simeq \bigoplus_{n \in \mathbb{Z}} e^{in\theta}$$

where each component is irreducible and two of them are inequivalent.

Since any $g = (a, t, \theta)$ can be written as $g = (0, t, 0)(t^2 a, 0, \theta)$, due to (K2) any function $F \in \mathcal{H} = \mathcal{H}_1$ can be identified with its restriction to $\mathbb{R}_+$. On the other hand (K3) becomes

$$\int_0^\infty |F(\sqrt{y})|^2 y^{-1} dy = \int_0^\infty |F(t)|^2 2t^{-1} dt < +\infty,$$

hence we have the following unitary identifications

$$\mathcal{H} \simeq L^2(\mathbb{R}_+, 2t^{-1}dt, L^2(S^1, d\xi/2)) \simeq L^2(\mathbb{R}_+ \times S^1, t^{-1} dt d\xi).$$

The unitary map $S : L^2(X) \to \mathcal{H}$ is given explicitly by

$$(Sf)(t, \xi) = t\, T_{t^2, t^{-1}}(f_{1,t^2})(\xi) = t f(t\cos\xi, t\sin\xi).$$

For any $n \in \mathbb{Z}$ the space $\mathcal{H}_n$ carrying the representation induced by $e^{2\pi i a} e^{in\theta}$ is

$$\mathcal{H}_n = \{F \in L^2(\mathbb{R}_+ \times \mathcal{T}, t^{-1} dt d\xi) \mid F(t, \xi) = F_n(t) e^{2\pi i n \xi} \text{ where } F_n \in L^2(\mathbb{R}_+, t^{-1}dt)\}.$$

Hence given $\eta \in L^2(X)$, since $S\eta = \sum_{n\in\mathbb{Z}} F_n e^{in\xi}$ with $F_n \in L^2(\mathbb{R}_+, t^{-1}dt)$, then η is an admissible vector if and only if, for any $n \in \mathbb{Z}$,

$$\int_0^{+\infty} \left(\int_{S^1} |F_n(\sqrt{y}) e^{in\xi}|^2 \frac{d\xi}{2} \right) y^{-2} dy = \frac{\dim \mathcal{K}_n}{\operatorname{vol} \mathcal{T}} = 2,$$

since $\dim \mathcal{K}_n = 1$. By the change of variable $t = \sqrt{y}$, this is equivalent to

$$\int_0^{+\infty} |F_n(t)|^2 t^{-3} dt = \frac{1}{\pi}.$$

Since

$$F_n(t) = \frac{1}{2\pi} \int_0^{2\pi} t\eta(t\cos\xi, t\sin\xi) e^{-in\xi} d\xi = t\hat{\eta}(t,n),$$

we have that the set of admissible vectors is the Lebesgue measurable functions $\eta : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\sum_{n\in\mathbb{Z}} \int_0^{+\infty} |\hat{\eta}(t,n)|^2 t dt < +\infty \iff \eta \in L^2(\mathbb{R}^2)$$
$$\int_0^{+\infty} |\hat{\eta}(t,n)|^2 t^{-1} dt = \frac{1}{\pi} \qquad \forall n \in \mathbb{Z}.$$

4.5. **Example 5.** As in the above example, Assumption 1 is satisfied with the choice of $X = \mathbb{R}^2 \setminus \{x_2 = 0\}$ and $Y = \Phi(X) = \mathbb{R} \setminus \{0\}$ since X is a H-invariant open set whose complement has zero Lebesgue measure. The group H acts freely on Y so that the orbit space Y/H reduces to a singleton and Assumption 2 holds true. We choose $y_0 = 1$ as the origin of the orbit so that the corresponding stabilizer is the non-compact group $\mathbb{R}^* = H_1$. To prove that G a reproducing group, we use Theorem 14.
The measure λ on Y/H is trivial, so that in Theorem 8 the relatively invariant measure on Y is $\tau_1 = dy$. Furthermore, the map

$$\xi \mapsto (\xi, 1)$$

is diffeomorphism of $\mathbb{R}$ onto the Riemannian submanifold $\Phi^{-1}(1) = \{x_2 = 1\}$. The Riemannian measure on $\mathbb{R}$ is $d\xi$ and $(J\Phi)(x) = 1$, so that (58) gives that, for all $f \in C_c(X)$

$$\int_X \varphi(x_1, x_2) d\nu_1(x_1, x_2) = \int_{\mathbb{R}} \varphi(\xi, 1) d\xi.$$

By setting $h(y) = (y, 0)$ so that $h(y)[1] = y$, (30) gives the Haar measure ds on $\mathbb{R}$ is db since

$$\int_H \varphi(t,b)|t| \frac{dt}{|t|} db = \int_{\mathbb{R}^*} \left(\int_{\mathbb{R}} \varphi(y,b)\, db \right) dy.$$

The representation Λ_1 of $\mathbb{R}$ on $L^2(X, \nu_1) \simeq L^2(\mathbb{R}, d\xi)$ is the regular representation, so that

$$L^2(X, \nu_1) \simeq \int_{\mathbb{R}} \mathbb{C}\, d\omega$$
$$\Lambda_{1,b} \simeq \int_{\mathbb{R}} e^{2\pi\omega b}\, d\omega$$

where each component is irreducible, two of them are inequivalent and the intertwining operator is given by the Fourier transform.

Since any $g = (a, t, b) \in G$ can be written as $g = (0, t, 0)(ta, 0, b)$, due to (K2) any function $F \in \mathcal{H} = \mathcal{H}_1$ can be identified with its restriction to $\mathbb{R}^*$ and we have the following unitary identifications

$$\mathcal{H} \simeq L^2(\mathbb{R}^*, t^{-1}dt, L^2(\Phi^{-1}(1), \nu_1)) \simeq L^2(\mathbb{R}^2, y^{-1}dyd\xi).$$

The unitary map $S : L^2(X) \to L^2(\mathbb{R}^2, y^{-1}dyd\xi)$. is given explicitly by

$$(Sf)(y, \xi) = |t|^{\frac{1}{2}} f(y, \xi).$$

Theorem 14 ensures that $\eta \in L^2(X)$ is an admissible vector if and only if for all $u \in L^2(\mathbb{R}, d\xi)$

$$\begin{aligned}\int_{\mathbb{R}} |u(\xi)|^2 \, d\xi &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\langle u, |y|^{-\frac{1}{2}} \Lambda_{1,b}(S\eta)(y, \cdot)\rangle|^2 \, db \right) |y|^{-1} dy \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\hat{u}(\omega)|^2 |\hat{\eta}(y, \omega)|^2 d\omega \right) |y|^{-1} dy\end{aligned}$$

where we use that $\Delta(h(y)) = \alpha(h(y))^{-1} = |y|$ and $\hat{}$ denotes the Fourier transform with respect to ξ. It follows that the set of admissible vectors is the set of Lebesgue measurable functions $\eta : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\begin{aligned}&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\hat{\eta}(y, \omega)|^2 d\omega \right) dy < +\infty \iff \eta \in L^2(\mathbb{R}^2) \\ &\int_{\mathbb{R}} |\hat{\eta}(y, \omega)|^2 |y|^{-1} dy = 1 \qquad \text{almost every } \omega \in \mathbb{R},\end{aligned}$$

whose existence is clear, for example take any strictly positive continuous function $\sigma \in L^1(\mathbb{R})$ and define

$$\widehat{\eta}(y, \omega) = \left(\frac{1}{\sqrt{2\pi}\sigma(\omega)} |y| e^{-\frac{y^2}{2\sigma(\omega)^2}} \right)^{\frac{1}{2}}.$$

5. Appendix: some measure theory revisited

In this Appendix we review some known facts that are somehow hard to locate in the literature in a way that is both easily accessible and stated under the assumptions that we are making. The spaces X and Y are as in Section 3 and are regarded as measure spaces with respect to the Lebesgue measure, denoted dx and dy respectively.

5.1. **Disintegration of measures.** We start by adapting to our setting some facts from integration theory on general locally compact spaces. The main reference for the issues at hand is [3]. Hereafter, $C_c(X)$ denotes the space of compactly supported functions on X, endowed with the locally convex (metrizable and separable) topology for which a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(X)$ converges to zero if there exists a compact set K such that $\operatorname{supp} \varphi_n \subset K$ for all n and $\lim_{n\to\infty} \sup_{x\in K} |\varphi_n(x)| = 0$. We denote by $M(X)$ the topological dual of $C_c(X)$; when equipped with the $\sigma(M(X), C_c(X))$-topology, the topological dual of $M(X)$ is again $C_c(X)$ ([23], Th. IV.20). Since X is second countable, the Riesz-Markov representation theorem uniquely identifies the measures with the positive elements of $M(X)$. By the word measure on a locally compact second countable topological space, we mean a positive measure defined on the Borel σ-algebra, which is finite on compact subsets.

The following theorem, in some sense a version of Fubini's theorem, summarizes the main properties of the kind of disintegration of measures we are concerned with. The main point here, though, is the possibility of extending the disintegration from C_c to L^1. We state it for X and Y, but it also holds *verbatim* if we replace X and Y with two arbitrary locally compact and second countable topological spaces.

THEOREM 25. *Suppose that ω is a measure on X and ρ a measure on Y and let $\Psi : X \to Y$ be a ω-measurable map. Assume further that $\{\omega_y\}$ is a family of measures on X such that*

(a) *ω_y is concentrated on $\Psi^{-1}(y)$ for all $y \in Y$;*

(b) $\displaystyle\int_X \varphi(x) d\omega(x) = \int_Y \left(\int_X \varphi(x) d\omega_y(x) \right) d\rho(y)$ *for all $\varphi \in C_c(X)$.*

Then, for any ω-measurable function $f : X \to \mathbb{C}$ the following facts hold true:

(i) *f is ω_y-measurable for almost every $y \in Y$;*

(ii) *f is ω-integrable if and only if $\displaystyle\int_Y \left(\int_X |f(x)| d\omega_y(x) \right) d\rho(y)$ is finite;*

(iii) *if f is ω-integrable, then f is ω_y-integrable for ρ-almost every $y \in Y$, the function (defined almost everywhere) $y \mapsto \int_X f(x) d\omega_y(x)$ is ρ-integrable, and*

$$\int_X f(x) d\omega(x) = \int_Y \left(\int_X f(x) d\omega_y(x) \right) d\rho(y); \tag{50}$$

(iv) *if $\{\omega'_y\}$ is another family of measures on X satisfying (a) and (b), then $\omega'_y = \omega_y$ for ρ-almost all $y \in Y$.*

Proof. The theorem is essentially contained in [3], scattered in several statements. For the proof of (i), (ii) and (iii) we quote from Chapter 5, and for the proof of (iv) from Chapter 6.

Statement (i) is the content of item a) Prop. 4, § 3.2, taking into account that, since it is second countable, X is σ-compact and, a fortiori, ω-moderated (a subset is ω-moderated if it is contained into the union of a countable sequence of compact subsets and a ω-negligible set).

As for (ii), since X is second countable, Prop. 2, § 3.1, ensures that the family $\int_X \varphi(x) d\omega_y(x)$ is ρ-adequate in the sense of Def. 1, § 3.1. The equivalence of the two conditions in (ii) is then the content of the Corollary at the end of § 3.2.

As for (iii), it is just Th. 1, § 3.3, observing that any function is ω-moderated since X is ω-moderated (a function is ω-moderated if it is null on the complement of a ω-moderated subset).

Finally, for (iv), by assumption $\int_Y \omega_y d\rho(y) = \int_Y \omega'_y d\rho(y)$, where the integral is a scalar integral of vector valued functions taking value in $M(X)$. Now Lemma 1, § 3.1

ensures that $C_c(X)$ has a countable subset which is dense[4] in $C_c(X)$ with respect to the $\sigma(C_c(X), M(X))$ topology, so that, by Remark 2 in §1.1, it is enough to show that for any $\varphi \in C_c(X)$ and for ρ-almost every $y \in Y$

$$\int_X \varphi(x) d\omega_y(x) = \int_X \varphi(x) d\omega'_y(x).$$

This is in turn equivalent to proving that

$$\int_Y \left(\int_X \varphi(x) d\omega_y(x) \right) \xi(y) d\rho(y) = \int_Y \left(\int_X \varphi(x) d\omega'_y(x) \right) \xi(y) d\rho(y) \tag{51}$$

holds for all $\varphi \in C_c(X)$ and $\xi \in C_c(Y)$. Fix then $\varphi \in C_c(X)$ and $\xi \in C_c(Y)$, and put $f(x) = \xi(\Psi(x))\varphi(x)$. This function is ω-measurable since Ψ is ω-measurable and ξ and φ are continuous, it is bounded since both ξ and φ are bounded, and it has a compact support since φ is compactly supported. Hence f is ω- integrable. Applying twice (50) we get

$$\int_Y \left(\int_X \xi(\Psi(x))\varphi(x) d\omega_y(x) \right) d\rho(y) = \int_Y \left(\int_X \xi(\Psi(x))\varphi(x) d\omega'_y(x) \right) d\rho(y). \tag{52}$$

Given $y \in Y$, (a) implies that $\xi(\Psi(x)) = \xi(y)$ for ω_y-almost all $x \in X$, so that

$$\int_Y \left(\int_X \xi(\Psi(x))\varphi(x) d\omega_y(x) \right) d\rho(y) = \int_Y \left(\int_X \varphi(x) d\omega_y(x) \right) \xi(y) d\rho(y),$$

and similarly for the right hand side of (52). Hence (51) is true and the claim is proved. □

The integral formula (b) will be written for short

$$d\omega = \int_Y \omega_y \, d\rho(y). \tag{53}$$

5.2. **Direct integrals.** Next we recall the definition of direct integral, following [13]. Hereafter we assume that the hypotheses of Theorem 25 are satisfied. Fix a countable family $\{\varphi_k\}_{k\in\mathbb{N}}$ dense in $C_c(X)$, and hence also in every $L^2(X, \omega_y)$, with $y \in Y$. The map $y \mapsto \langle \varphi_k, \varphi_\ell \rangle_{\omega_y}$ is ρ-measurable since it is ρ-integrable by the hypothesis (a). Under these circumstances, $\{\varphi_k\}_{k\in\mathbb{N}}$ is called a measurable structure for the family of Hilbert spaces $\{L^2(X, \omega_y)\}$. The direct integral $\int_Y L^2(X, \omega_y) dy$ is defined as the set consisting of all the families $\{f_y\}$ satisfying:

- (D1) $f_y \in L^2(X, \omega_y)$ for all $y \in Y$;
- (D2) $\int_Y \|f_y\|^2_{\omega_y} d\rho(y) < +\infty$;
- (D3) $y \mapsto \langle f_y, \varphi_k \rangle_{\omega_y}$ is ρ-measurable for all k.

Two families $\mathcal{F} = \{f_y\}$ and $\mathcal{G} = \{g_y\}$ are identified if for almost every $y \in Y$ $f_y = g_y$ as elements in $L^2(X, \omega_y)$. The space $\int_Y L^2(X, \omega_y) d\rho(y)$ is a Hilbert space under

$$\langle \mathcal{F}, \mathcal{G} \rangle = \int_Y \langle f_y, g_y \rangle_{\omega_y} d\rho(y).$$

[4] It is proved there that there exists a countable subset $S \subset C_c(X)$ such that for every $f \in C_c(X)$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ in S converging to f uniformly and $|f_n| \leq |f|$ for all n.

Since $C_c(X)$ has a dense countable subset, (see footnote 4), (D3) is equivalent to

(D3') $y \mapsto \langle f_y, \varphi\rangle_{\omega_y}$ is ρ-measurable for all $\varphi \in C_c(X)$,

so that, as long as we choose the functions of $\{\varphi_k\}_{k\in\mathbb{N}}$ in $C_c(X)$, the measurable structure is independent of the choice of the particular family.

PROPOSITION 26. *Given $f \in L^2(X,\omega)$, there exists a unique family $\{f_y\}$ in the Hilbert space direct integral $\int_Y L^2(X,\omega_y)d\rho(y)$ such that, for almost every $y \in Y$, the equality $f_y(x) = f(x)$ holds for ω_y-almost every $x \in X$. Furthermore, the map $f \mapsto \{f_y\}$ is a unitary operator from $L^2(X,\omega)$ onto $\int_Y L^2(X,\omega_y)d\rho(y)$.*

Proof. By hypothesis (b) of Theorem 25, for every $\varphi \in C_c(X)$ we have

$$\int_X \varphi(x)d\omega(x) = \int_Y \left(\int_X \varphi(x)d\omega_y\right) d\rho(y).$$

Given a function[5] $f : X \to \mathbb{C}$ which is square-integrable with respect to ω, hence in particular ω-measurable, (i) of Theorem 25 implies that f is ω_y-measurable for almost every $y \in Y$. Further, since $|f|^2$ is integrable with respect to ω, (iii) of the same theorem ensures that $|f|^2$ is ω_y-integrable for almost all $y \in Y$, the map $y \mapsto \int_X |f(x)|^2 d\omega_y(x)$ is integrable, and

$$\int_X |f(x)|^2\, d\omega(x) = \int_Y \left(\int_X |f(x)|^2 d\omega_y(x)\right) d\rho(y). \tag{54}$$

Hence there is a ρ-full set $Y' \subset Y$ such that, if $y \in Y'$, f is square-integrable with respect to ω_y. For $y \in Y'$ define f_y to be the equivalence class of f in $L^2(X,\omega_y)$ and, for $y \notin Y'$, put $f_y = 0$.

We claim that $\mathcal{F} = \{f_y\}$ is in $\int_Y L^2(X,\omega_y)d\rho(y)$. By (54), conditions (D1) and (D2) are clearly satisfied. To prove (D3'), take $\varphi \in C_c(X)$. Clearly, $f\overline{\varphi}$ is ω-integrable and hence, by (iii) of Theorem 25, it is ω_y-integrable for almost every $y \in Y$ and

$$y \mapsto \int_X f(x)\overline{\varphi(x)}d\omega_y(x) = \langle f_y, \varphi\rangle_{\omega_y}$$

is integrable, hence measurable. Therefore $f \mapsto \mathcal{F}$ is a well defined map from the space of square-integrable functions on X to $\int_Y L^2(X,\omega_y)d\rho(y)$, it is linear and, by (54),

$$\int_X |f(x)|^2 d\omega(x) = \int_Y \|f_y\|^2_{\omega_y} d\rho(y). \tag{55}$$

Hence, it defines an isometry from $L^2(X,\omega)$ into $\int_Y L^2(X,\omega_y)d\rho(y)$ and, by construction, for almost every $y \in Y$, the equality $f_y(x) = f(x)$ holds for ω_y-almost every $x \in X$.

We claim that the isometry $f \mapsto \mathcal{F}$ is surjective. It is enough to prove that for any family $\mathcal{F}$ whose members f_y are positive, there exists a positive $f \in L^2(X,\omega)$ such that, for almost every $y \in Y$, the equality $f_y(x) = f(x)$ holds for ω_y-almost every $x \in X$. Take then such an $\mathcal{F}$. First of all, we show that the family of measures $\{f_y \cdot \omega_y\}$ is scalarly integrable with respect to ρ. This is equivalent to saying that for all $\varphi \in C_c(X)$ the function $y \mapsto F_\varphi(y) = \int_X \varphi(x)f_y(x)d\omega_y(x)$, certainly well

[5] Here it is important that f is a function, and not an equivalence class modulo a.e. equality.

defined because (D1) implies that φf_y is ω_y-integrable for every $y \in Y$, is ρ-integrable. Indeed, (D3') says that F_φ is ρ-measurable, whereas Hölder's inequality and Cauchy-Schwartz give

$$\int_Y |F_\varphi(y)| d\rho(y) \leq \int_Y \|\varphi\|_{\omega_y} \|f\|_{\omega_y} d\rho(y) \leq \left(\int_Y \|\varphi\|^2_{\omega_y} d\rho(y) \right)^{1/2} \left(\int_Y \|f\|^2_{\omega_y} d\rho(y) \right)^{1/2}$$

so that by (D2) and (55) applied to φ yield

$$\int_Y |F_\varphi(y)| d\rho(y) \leq C \|\varphi\| < +\infty.$$

Hence the claim is proved and $\mu = \int_Y (f_y \cdot \omega_y) d\rho(y)$ defines a measure. We show next that μ is a measure with base[6] ω. This will produce the required f that maps to $\mathcal{F}$. The Lebesgue-Nikodym theorem (see Th. 2 ,§ 5.5, Ch. 5 of [3]) ensures that it is enough to prove that any compact subset $K \subset X$ for which $\omega(K) = 0$ satisfies $\mu(K) = 0$. Take such a K. Item (iii) of Theorem 25 applied to the characteristic function χ_K gives that for almost every $y \in Y$, K is ω_y-negligible and, *a fortiori*, $f_y \cdot \omega_y$-negligible. Thus, (50) with $\omega = \mu$, $\omega_y = f \cdot \omega_y$ and $f = \chi_K$ yields

$$\mu(K) = \int_Y \left(\int_K f_y(x) d\omega_y(x) \right) d\rho(y) = 0.$$

Hence there exists a locally integrable positive function f such that $f \cdot \omega = \mu$. Moreover, if $\varphi \in C_c(X)$, φf is integrable, so that again (iii) of Theorem 25 tells us that, for almost every $y \in Y$, φf is ω_y-integrable, the map $y \mapsto \int_X \varphi(x) f(x)\, d\omega_y(x)$ is integrable and by definition of μ

$$\int_Y \left(\int_X \varphi(x) f_y(x)\, d\omega_y(x) \right) d\rho(y) = \int_X \varphi(x)\, d\mu(x) = \int_Y \left(\int_X \varphi(x) f(x)\, d\omega_y(x) \right) d\rho(y).$$

By the above equality, (iv) of Theorem 25 may be applied to infer that for almost every $y \in Y$ the equality $f = f_y$ holds ω_y-almost everywhere. Finally, (D2) gives

$$\int_Y \left(\int_X |f(x)|^2 d\omega_y(x) \right) d\rho(y) = \int_Y \left(\int_X |f_y(x)|^2 d\omega_y(x) \right) d\rho(y) < +\infty.$$

Hence (iii) of Theorem 25 implies that f is square integrable. The equivalence class of f in $L^2(X, \omega)$ is then the element required to prove surjectivity. □

Both $L^2(X, \omega)$ and each of the spaces $L^2(X, \omega_y)$ can be identified with subspaces of $M(X)$, simply by viewing their elements as continuous linear functionals on $C_c(X)$ via integration with respect to ω and ω_y, respectively. Further, (iv) of Theorem 25 implies that saying that for almost every $y \in Y$ the equality $f_y(x) = f(x)$ holds for ω_y-almost every $x \in X$ is equivalent to

$$f \cdot \omega = \int_Y (f_y \cdot \omega_y)\, d\rho(y),$$

[6]A measure which is the product $\psi \cdot \mathcal{L}$ of a measure $\mathcal{L}$ by a locally $\mathcal{L}$-integrable positive function ψ is called a measure with base $\mathcal{L}$ (see Def. 2, § 5.2, Ch. V in [3]).

in the sense that the map $Y \to M(X)$, $y \mapsto f_y \cdot \omega_y$ is ρ-scalarly-integrable.[7] These remarks together with Proposition 26 imply that

$$L^2(X,\omega) = \int_Y L^2(X,\omega_y) d\rho(y) \tag{56}$$

by means of the equality in $M(X)$

$$f = \int_Y f_y d\rho(y), \tag{57}$$

where the integral is a scalar integral.

5.3. **The coarea formula for submersions.** Below we give a simple proof of the Coarea Formula for submersions; the general case is due to Federer [12]. Suppose that $n \leq d$ and let $X \subset \mathbb{R}^d$ be an open set. Recall that a C^1-map $\Phi : X \to \mathbb{R}^n$ is called a submersion if its differential Φ_{*x} is surjective for all $x \in X$. For every $y \in Y = \Phi(X)$, let $dv^y(x)$ denote the volume element of the Riemannian submanifold $\Phi^{-1}(y)$ and by $J\Phi$ the Jacobian. We introduce the measure ν_y on X by

$$\nu_y(E) = \int_{\Phi^{-1}(y)\cap E} \frac{dv^y(x)}{(J\Phi)(x)}, \qquad E \in \mathcal{B}(X). \tag{58}$$

It is worth observing that ν_y is finite on compact sets and concentrated on $\Phi^{-1}(y)$.

Theorem 27 (Coarea formula for submersions). *Suppose that $\Phi : X \to \mathbb{R}^n$ is a submersion. Then*

$$dx = \int_Y d\nu_y \, dy, \tag{59}$$

where dx and dy are the Lebesgue measures on $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively.

Proof. We must show that

$$\int_X f(x)\, dx = \int_Y \left(\int_X f(x) \frac{dv^y(x)}{(J\Phi)(x)} \right) dy$$

holds for every $f \in C_c(X)$. Fix $x_0 \in X$. Since Φ_{*x_0} is surjective, the Inverse Mapping Theorem implies (Corollary 5.8 in [20]) that there exists a diffeomorphism $\Psi : U \times V \mapsto W$ such that

$$\Phi(\Psi(z,y)) = y \qquad z \in U,\, y \in V, \tag{60}$$

where U is an open subset of $\mathbb{R}^{d-n}$, V is an open subset of $\mathbb{R}^n$ and W is an open neighborhood of x_0.

Take $f \in C_c(X)$. For any such f, since $\operatorname{supp} f$ is compact, by choosing a suitable finite covering if necessary, we can always assume that $\operatorname{supp} f \subset W$. The change of variables formula and Fubini Theorem give

$$\int_W f(x)\, dx = \int_V \left(\int_U f(\Psi(z,y))(J\Psi)(z,y)\, dz \right) dy. \tag{61}$$

[7] This means, by definition, that for any $\varphi \in C_c(X)$ the function $y \mapsto \int_X \varphi(x) f_y(x)\, d\omega_y(x)$ is integrable with respect to ρ.

To obtain the coarea formula we simply compute the Jacobian $J\Psi$. Observe that for any given $y \in V$, $\Psi^y = \Psi(\cdot, y)$ is a diffeomorphism from U onto $W \cap \Phi^{-1}(y)$, regarded as a submanifold. In particular, using this local chart, the volume element at the point $x = \Psi(z, y)$ is given by

$$dv^y(x) = \sqrt{\det\left[{}^t(\Psi^y)_{*z}(\Psi^y)_{*z}\right]}\, dz. \tag{62}$$

Taking the derivatives of(60) with respect to z and y separately, we obtain

$$\Phi_{*\Psi(z,y)}\, D_1\Psi_{(z,y)} = 0, \qquad \Phi_{*\Psi(z,y)}\, D_2\Psi_{(z,y)} = I_{n\times n}. \tag{63}$$

Fix $(z, y) \in U \times V$ and let P_1 denote the orthogonal projection from $\mathbb{R}^d$ onto $\ker \Phi_{*\Psi(z,y)}$, and $P_2 = I - P_1$ the orthogonal projection onto $[\ker \Phi_{*\Psi(z,y)}]^\perp$, which is a subspace of dimension n because Φ is a submersion. From (63) it follows that

$$P_2(D_1\Psi)_{(z,y)} = 0, \qquad P_2(D_2\Psi)_{(z,y)} = (\Phi_{*\Psi(z,y)} \circ \iota)^{-1}, \tag{64}$$

where $\iota : [\ker \Phi_{*\Psi(z,y)}]^\perp \to \mathbb{R}^d$ is the natural injection. Let $R \in \mathrm{O}(d)$ be the rotation that takes $\ker \Phi_{*\Psi(z,y)}$ onto the z-hyperplane (first $d-n$ coordinates) and its orthogonal complement onto the y-hyperplane (last n coordinates), so that $RP_1(z, y) = z$ and $RP_2(z, y) = y$. Then (64) imply

$$R\Psi_{*(z,y)} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where $A = R(D_1\Psi)_{(z,y)}$, $B = RP_1(D_2\Psi)_{(z,y)}$ and $C = RP_2(D_2\Psi)_{(z,y)}$. Therefore

$$(J\Psi)(z, y) = |\det R\Psi_{*(z,y)}| = |\det A|\,|\det C| = \frac{\sqrt{\det\left[{}^t(\Psi^y)_{*z}(\Psi^y)_{*z}\right]}}{\sqrt{\det\left[\Phi_{*\Psi(z,y)}{}^t\Phi_{*\Psi(z,y)}\right]}},$$

where we have used (64). Taking (62) into account, for $x = \Psi(z, y)$ we have

$$(J\Psi)(z, y)\, dz = \frac{dv^y(x)}{(J\Phi)(x)},$$

which inserted in (61) yields the result. □

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